Int. J. Solids Structures, 1977, Vol. 13, pp. 747–769. Pergamon Press. Printed in Great Britain

AXISYMMETRIC THERMAL STRESS IN A THIN SPHERICAL SHELL BY THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

HARRY E. WILLIAMS
Harvey Mudd College, Claremont, CA 91711, U.S.A.

(Received 21 October 1976)

Abstract—A solution of the equations of linear Thermoelasticity is presented for a closed shell with constant material properties. The solution is constructed by matching asymptotic expansions in the thinness parameter (h/a = thickness/radius of curvature) in the various regions of the shell. For clamped conditions at the (meridional opening angle-constant) edge ($\theta = \theta_0$), the solution has the character expected of a thin shell, i.e. a membrane region in the interior with a "thin shell" boundary layer near $\theta = \theta_0$. For the stress-free condition, however, an "Elasticity" layer of meridional width of order h must be introduced between the "thin-shell" layer and the edge ($\theta = \theta_0$). This solution is also compared with an asymptotic solution of the thin-shell equations and shown to agree through two orders of magnitude of $(h/a)^{1/2}$.

INTRODUCTION

As is well known, it is not possible to obtain a general solution of the equations of linear Thermoelasticity for a spherical shell containing an axisymmetric temperature distribution. The problem lies in the inability of all analytical methods to satisfy general boundary conditions simultaneously on the inner and outer surfaces ($r = a, a + h$) and the edge surface ($\theta = \theta_0$). However, if a numerical solution for a given temperature distribution is all that is desired, this can readily be obtained using, for example, one of the computer programs based on the Finite Element method.

Alternatively, one might ask if the same information might be obtained from a solution of the thin-shell equations. As is well known, analytical solutions are readily obtained for these equations as they require boundary conditions to be satisfied only on the boundary $\theta = \theta_0$. The remaining stress boundary conditions on the inner and outer surfaces are included in the equations of equilibrium as loading terms. It should be noted, however, that the principal advantage of replacing the equations of Elasticity with the thin-shell equations is that numerical solutions are less expensive to obtain.

The question then facing the analyst is whether the stress distribution given by the thin-shell equations (numerically or analytically) is an accurate representation of that which would be obtained (numerically) from the equations of Elasticity. If the shell is thin, it is reasonable to expect that such would be the case. It is the principal aim of this paper to show that the thin-shell equations are indeed an adequate representation for spherical shells that are sufficiently thin. We intend to accomplish this by obtaining an approximate solution of the equations of Thermoelasticity and showing that it is identical to a solution of the thin-shell equations to the same order of approximation.

The solution of the equations of Thermoelasticity is obtained for thin spherical shells by the method of matched asymptotic expansions in the manner proposed by Cole[1]. It should be noted that the essential structure of the solution was anticipated earlier by Johnson and Reissner[2] in their study of cylindrical shells. We propose here to incorporate the order of magnitude analysis used by both Cole[1] and Johnson and Reissner[2] with the method of matching proposed by Cole to construct the solution for a heated, thin spherical shell that is either clamped or stress-free at the edge $\theta = \theta_0$. As was observed by both[1, 2], we find that the solution behaves differently throughout the shell, being characterized by length scales (see Fig. 1) of the order a, $\sqrt{(ah)}$, and h in the "membrane", "thin-shell" layer and "Elasticity" layer respectively. In all three regions, the solution is expressed as an asymptotic series in the thinness parameter ($\bar{h} = h/a$), and a hierarchy of equations is obtained through a limiting procedure which governs the approximation in each region. The boundary conditions that must be satisfied by the solution in each region are obtained, following Cole[1], by requiring that the

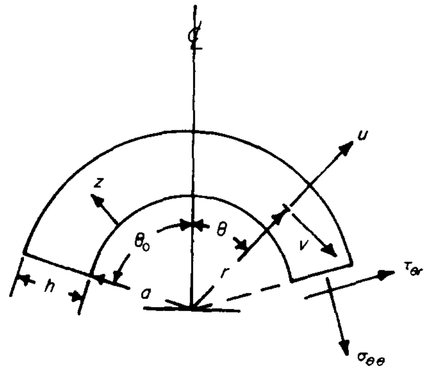

Fig. 1. Nomenclature.

series representations in adjacent regions match asymptotically in a region intermediate to each pair of regions.

As will be shown subsequently, the solution adjacent to a stress-free edge is governed by an equation identical to that obtained and solved numerically in [3] for the thermal stress distribution adjacent to the edge of a thin, circular disk. As the details of the numerical scheme were described in [3], we will not repeat the procedure here for the sake of brevity, and will refer the reader to [3] whenever necessary for further details.

The solution of the thin-shell equations that will be referred to here for comparison has been presented in [4]. The method of solution was essentially to apply the order of magnitude analysis used by Wittrick[5] to Reissner's[6] equations, and use the method of matching proposed by Cole[1] to evaluate the constants of integration. Formulae are also presented for the meridional and circumferential stress components that are correct to order $(h/a)^{1/2}$ compared to unity using formulae proposed by Kraus[7].

With the establishment of solutions to both sets of equations, it will be shown that they are identical through two orders of magnitude of the parameter $(h/a)^{1/2}$. Thus, a solution for all the stresses and displacements accompanying a general axisymmetric temperature distribution has been obtained and shown to be valid for sufficiently thin spherical shells. This is not only an important result in itself, as such formulae are always valuable for design purposes, but it has important implications in other areas. Though it can only be suggested, the above results indicate that it is likely that similar results might also be obtained for general dome-like shells under both axisymmetric and slowly varying circumferential temperature distributions. This has the important consequence that a designer could reasonably choose to evaluate the design of a thin shell by means of a relatively simple Finite Element solution of the thin-shell equations rather than require a more costly solution of the equations of Elasticity.

DIMENSIONLESS FORM OF THE GOVERNING EQUATIONS

The equations governing the stresses and displacements in a spherical shell due to an axisymmetric temperature distribution are given in Boley and Weiner[8] for a linear, isotropic material undergoing small displacements. If we define a dimensionless temperature distribution (Θ) in terms of the actual temperature distribution (T) and a temperature scale (T_0) by

$$T - T_{\text{REF}} = T_0\Theta(\theta, \bar{z})$$

and introduce the following dimensionless stress and displacement variables

$$(u, v) = \alpha a T_0(\bar{u}, \bar{v})$$

$$(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{\phi\phi}, \tau_{r\theta}) = \alpha E T_0(\bar{\sigma}_r, \bar{\sigma}_\theta, \bar{\sigma}_\phi, \bar{\tau})$$

into the governing equations, there results the two equations of equilibrium

$$\frac{\partial \bar{\sigma}_r}{\partial \bar{z}} + \frac{\bar{h}}{1 + \bar{h}\bar{z}}\left(\frac{\partial \bar{\tau}}{\partial \theta} + 2\bar{\sigma}_r - \bar{\sigma}_\theta - \bar{\sigma}_\phi + \bar{\tau}\cot\theta\right) = 0 \tag{1}$$

$$\frac{\partial \bar{\tau}}{\partial \bar{z}}+\frac{\bar{h}}{1+\bar{h}\bar{z}}\left[\frac{\partial \bar{\sigma}_\theta}{\partial \theta}+(\bar{\sigma}_\theta-\bar{\sigma}_\phi)\cot\theta+3\bar{\tau}\right]=0 \tag{2}$$

and the four stress-displacement equations

$$\frac{\partial \bar{u}}{\partial \bar{z}}=\bar{h}[\bar{\sigma}_r-\nu(\bar{\sigma}_\theta+\bar{\sigma}_\phi)+\Theta] \tag{3}$$

$$\bar{u}+\frac{\partial \bar{v}}{\partial \theta}=(1+\bar{h}\bar{z})[\bar{\sigma}_\theta-\nu(\bar{\sigma}_r+\bar{\sigma}_\phi)+\Theta] \tag{4}$$

$$\bar{u}+\bar{v}\cot\theta=(1+\bar{h}\bar{z})[\bar{\sigma}_\phi-\nu(\bar{\sigma}_\theta+\bar{\sigma}_r)+\Theta] \tag{5}$$

$$\frac{\partial \bar{v}}{\partial \bar{z}}+\frac{\bar{h}}{1+\bar{h}\bar{z}}\left(\frac{\partial \bar{u}}{\partial \theta}-\bar{v}\right)=2(1+\nu)\bar{h}\bar{\tau}. \tag{6}$$

The radial coordinate (r) has been replaced by its dimensionless counterpart ($\bar{z}$) given by

$$r=a(1+\bar{h}\bar{z}) \qquad (0\leq \bar{z}\leq 1)$$

and $\bar{h}(=h/a)$ is the thinness parameter. The material properties are characterized by Young's modulus (E), the coefficient of thermal expansion (α) and Poisson's ratio (ν). In the sections which follow, we will seek solutions to these equations for thin shells that are heated only, i.e. satisfy the stress-free boundary conditions on the inner and outer surfaces ($\bar{z}=0,1$) that

$$\sigma_{rr},\ \tau_{r\theta}=0 \qquad (0\leq\theta\leq\theta_0).$$

The boundary conditions on the outer edge surface ($\theta=\theta_0$) will be chosen from either the stress-free condition that

$$\sigma_{\theta\theta},\ \tau_{r\theta}=0 \qquad (0\leq \bar{z}\leq 1)$$

or the clamped condition

$$u,\ v=0 \qquad (0\leq \bar{z}\leq 1).$$

LIMITING SOLUTION IN THE INTERIOR

The method used here for constructing solutions for thin shells is motivated by the solution proposed by Cole[1] for a similar problem. The essential feature of this method is the observation that the stress and displacement variables can have different orders of magnitude in different regions of the shell as $\bar{h}\to 0$. In particular, for the interior region of the shell, if we assume that the solution has the following form as $\bar{h}\to 0$

$$\begin{aligned}
\bar{u}(\theta,\bar{z},\bar{h})&=\bar{u}_0(\theta,\bar{z})+\bar{h}^{1/2}\bar{u}_1(\theta,\bar{z})+0(\bar{h})\\
\bar{v}(\theta,\bar{z},\bar{h})&=\bar{h}^{1/2}[\bar{v}_0(\theta,\bar{z})+\bar{h}^{1/2}\bar{v}_1(\theta,\bar{z})+0(\bar{h})]\\
\bar{\sigma}_r(\theta,\bar{z},\bar{h})&=\bar{h}[\bar{\sigma}_r^{(0)}(\theta,\bar{z})+\bar{h}^{1/2}\bar{\sigma}_r^{(1)}(\theta,\bar{z})=0\bar{h})]\\
\bar{\tau}(\theta,\bar{z},\bar{h})&=\bar{h}[\bar{\tau}_0(\theta,\bar{z})+\bar{h}^{1/2}\bar{\tau}_1(\theta,\bar{z})+0(\bar{h})]\\
\bar{\sigma}_\theta(\theta,\bar{z},\bar{h})&=\bar{\sigma}_\theta^{(0)}(\theta,\bar{z})+\bar{h}^{1/2}\bar{\sigma}_\theta^{(1)}(\theta,\bar{z})+0(\bar{h})\\
\bar{\sigma}_\phi(\theta,\bar{z},\bar{h})&=\bar{\sigma}_\phi^{(0)}(\theta,\bar{z})+\bar{h}^{1/2}\bar{\sigma}_\phi^{(1)}(\theta,\bar{z})+0(\bar{h})
\end{aligned}$$

it follows from the governing equations (1)–(6), when these forms are substituted and repeated limits $\bar{h}\to 0$ are taken, that the following hierarchy of equations must be satisfied:

$$\frac{\partial \bar{\sigma}_r^{(i)}}{\partial \bar{z}}-\bar{\sigma}_\theta^{(i)}-\bar{\sigma}_\phi^{(i)}=0 \qquad (i=0,1) \tag{7}$$

$$\frac{\partial \bar{\tau}_{(i)}}{\partial \bar{z}} + \frac{\partial \bar{\sigma}_\theta^{(i)}}{\partial \theta} + (\bar{\sigma}_\theta^{(i)} - \bar{\sigma}_\phi^{(i)}) \cot \theta = 0 \qquad (i = 0, 1) \tag{8}$$

$$\frac{\partial \bar{u}_{(i)}}{\partial \bar{z}} = 0 \qquad (i = 0, 1) \tag{9}$$

$$\bar{u}_0 = \Theta + \bar{\sigma}_\theta^{(0)} - \nu \bar{\sigma}_\phi^{(0)} \tag{10}$$

$$\bar{u}_1 + \frac{\partial \bar{v}_0}{\partial \theta} = \bar{\sigma}_\theta^{(1)} - \nu \bar{\sigma}_\phi^{(1)} \tag{11}$$

$$\bar{u}_0 = \Theta + \bar{\sigma}_\phi^{(0)} - \nu \bar{\sigma}_\theta^{(0)} \tag{12}$$

$$\bar{u}_1 + \bar{v}_0 \cot \theta = \bar{\sigma}_\phi^{(1)} - \nu \bar{\sigma}_\theta^{(1)} \tag{13}$$

$$\frac{\partial \bar{v}_0}{\partial \bar{z}} = 0 \tag{14}$$

$$\frac{\partial \bar{v}_1}{\partial \bar{z}} + \frac{\partial \bar{u}_0}{\partial \theta} = 0. \tag{15}$$

The boundary conditions on the inner, outer surfaces accompanying this assumed form are

$$\bar{\sigma}_r^{(0)} = \bar{\sigma}_r^{(1)} = \cdots = 0 \qquad \bar{\tau}_0 = \bar{\tau}_1 = \cdots = 0 \qquad (\bar{z} = 0, 1).$$

The solution of the first order equations can readily be found, and is expressed in terms of the temperature integrals $\Theta_m(\theta)$, $\Xi(\theta, \bar{z})$ where

$$\Theta_m(\theta) = \int_0^1 \mathrm{d}x \Theta(\theta, x) \qquad \Xi(\theta, \bar{z}) = \int_0^{\bar{z}} \Theta(\theta, x)\, \mathrm{d}x \qquad \Xi_m(\theta) = \int_0^1 \mathrm{d}x \Xi(\theta, x).$$

The first step in the solution process is the observation that the radial normal strain vanishes eqn (9) identically. This requires that

$$\bar{u}_0 = \bar{U}_0(\theta)$$

and allows us to explicitly determine that $\bar{z}$-dependency of the meridional and circumferential stresses from the associated strain components eqns (10) and (12). Thus, with

$$\bar{\sigma}_\theta^{(0)} = \bar{\sigma}_\phi^{(0)} = (\bar{U}_0 - \Theta)/(1 - \nu) \tag{16}$$

and the stress-free boundary conditions on the inner surface, the equations of equilibrium eqns (7) and (8) can be integrated with respect to $\bar{z}$ to yield the following expressions for the radial and shear stresses

$$\bar{\sigma}_r^{(0)} = 2[\bar{z}\bar{U}_0(\theta) - \Xi(\theta, \bar{z})]/(1 - \nu) \tag{17}$$

$$\bar{\tau}_0 = \frac{\partial}{\partial \theta} [\Xi(\theta, \bar{z}) - \bar{z}\bar{U}_0(\theta)]/(1 - \nu). \tag{18}$$

The equations for the determination of $\bar{U}_0(\theta)$ follow from these expressions by requiring that the stress-free boundary conditions be satisfied on the outer surface ($\bar{z} = 1$). Both equations are satisfied by taking

$$\bar{U}_0(\theta) = \Theta_m(\theta). \tag{19}$$

The tangential displacement component $\bar{v}_0$ cannot be determined at this stage except to note from the requirement that the shear strain vanish eqn (14) that

$$\bar{v}_0 = \bar{V}_0(\theta).$$

The solution of the second order equations proceeds in a similar fashion. We again observe from the vanishing of the radial normal strain eqn (9) that

$$\bar{u}_1 = \bar{U}_1(\theta).$$

The accompanying meridional, circumferential stress components then follow from the associated strain components eqns (10) and (12) and take the form

$$(1-\nu^2)(\bar{\sigma}_\theta^{(1)}, \bar{\sigma}_\phi^{(1)}) = (1+\nu)\bar{U}_1 + (1, \nu)\frac{\mathrm{d}\bar{V}_0}{\mathrm{d}\theta} + (\nu, 1)\bar{V}_0 \cot\theta.$$

Noting that these stress components are independent of $\bar{z}$, we obtain the radial and shear stress components, as before, by integrating the equations of equilibrium eqns (7) and (8) with respect to $\bar{z}$ to obtain

$$\bar{\sigma}_r^{(1)} = \bar{z}(\bar{\sigma}_\theta^{(1)} + \bar{\sigma}_\phi^{(1)})$$

$$\bar{\tau}_1 = -\bar{z}\left[\frac{\mathrm{d}\bar{\sigma}_\theta^{(1)}}{\mathrm{d}\theta} + (\bar{\sigma}_\theta^{(1)} - \bar{\sigma}_\phi^{(1)})\cot\theta\right].$$

The equations for the determination of the displacements $\bar{U}_1$, $\bar{V}_0$ follow from these expressions and the requirement that the stresses vanish on the outer surface ($\bar{z} = 1$). However, before we determine the displacements, we find that the boundary conditions require that $\bar{\sigma}_\phi^{(1)} = -\bar{\sigma}_\theta^{(1)}$ where $\bar{\sigma}_\theta^{(1)}$ is the solution of

$$\frac{\mathrm{d}\bar{\sigma}_\theta^{(1)}}{\mathrm{d}\theta} + 2\bar{\sigma}_\theta^{(1)} \cot\theta = 0.$$

As the only solution of this equation is singular at $\theta = 0$, it follows that the solution of the second order equations that is regular at $\theta = 0$ is stress-free but allows the displacements

$$\bar{U}_1 = -A_0 \cos\theta \qquad \bar{V}_0 = A_0 \sin\theta \tag{20}$$

which can be identified as a rigid body translation.

Thus, the solution at this stage in the interior region of the shell is known to order $\bar{h}$ in comparison to unity save for an undertermined rigid body translation. In particular, the radial displacement and the meridional and shear stress components are determined completely. This has the important consequence that this solution is unable to satisfy either the stress-free or the clamped boundary conditions at the outer edge ($\theta = \theta_0$). It is apparent, then, that this solution is not valid in the neighborhood of the edge, and a boundary layer solution must be imposed if any boundary conditions are to be satisfied. In the following section, the equations governing the stress and displacement distributions in the boundary layer will be developed and solutions proposed for boundary conditions that can be satisfied.

THIN-SHELL BOUNDARY LAYER EQUATIONS

From the results of the previous section, it is apparent that some form of boundary layer must exist in the region near the edge. Following the solution proposed by Cole, we propose to investigate the behavior of the solution in this region by introducing the thin-shell boundary layer coordinate $\tilde{\theta}$, where

$$\theta = \theta_0 + \bar{h}^{1/2}\tilde{\theta}$$

and looking for a solution in the following form as $\bar{h} \to 0$

$$\bar{u}(\tilde{\theta}, \bar{z}, \bar{h}) = \tilde{u}_0(\tilde{\theta}, \bar{z}) + \bar{h}^{1/2}\tilde{u}_1(\tilde{\theta}, \bar{z}) + 0(\bar{h})$$

$$\bar{v}(\tilde{\theta}, \bar{z}, \bar{h}) = \bar{h}^{1/2}[\tilde{v}_0(\tilde{\theta}, \bar{z}) + \bar{h}^{1/2}\tilde{v}_1(\tilde{\theta}, \bar{z}) + 0(\bar{h})]$$

$$\tilde{\sigma}_r(\tilde{\theta}, \bar{z}, \bar{h}) = \bar{h}[\tilde{\sigma}_r^{(0)}(\tilde{\theta}, \bar{z}) + \bar{h}^{1/2}\tilde{\sigma}_r^{(1)}(\tilde{\theta}, \bar{z}) + 0(\bar{h})]$$
$$\tilde{\tau}(\tilde{\theta}, \bar{z}, \bar{h}) = \bar{h}^{1/2}[\tilde{\tau}_0(\tilde{\theta}, \bar{z}) + \bar{h}^{1/2}\tilde{\tau}_1(\tilde{\theta}, \bar{z}) + 0(\bar{h})]$$
$$\tilde{\sigma}_\theta(\tilde{\theta}, \bar{z}, \bar{h}) = \tilde{\sigma}_\theta^{(0)}(\tilde{\theta}, \bar{z}) + \bar{h}^{1/2}\tilde{\sigma}_\theta^{(1)}(\tilde{\theta}, \bar{z}) + 0(\bar{h})$$
$$\tilde{\sigma}_\Phi(\tilde{\theta}, \bar{z}, \bar{h}) = \tilde{\sigma}_\Phi^{(0)}(\tilde{\theta}, \bar{z}) + \bar{h}^{1/2}\tilde{\sigma}_\Phi^{(1)}(\tilde{\theta}, \bar{z}) + 0(\bar{h}).$$

When this form is substituted into the governing eqns (1)–(6) and repeated limits $\bar{h} \to 0$ are taken, we find that the following equivalent hierarchy of equations govern the solution

$$\frac{\partial \tilde{\sigma}_r^{(0)}}{\partial \bar{z}} + \frac{\partial \tilde{\tau}_0}{\partial \tilde{\theta}} - \tilde{\sigma}_\theta^{(0)} - \tilde{\sigma}_\Phi^{(0)} = 0 \tag{21}$$

$$\frac{\partial \tilde{\sigma}_r^{(1)}}{\partial \bar{z}} + \frac{\partial \tilde{\tau}_1}{\partial \tilde{\theta}} - \tilde{\sigma}_\theta^{(1)} - \tilde{\sigma}_\Phi^{(1)} + \tilde{\tau}_0 \cot \theta_0 = 0 \tag{22}$$

$$\frac{\partial \tilde{\tau}_0}{\partial \bar{z}} + \frac{\partial \tilde{\sigma}_\theta^{(0)}}{\partial \tilde{\theta}} = 0 \tag{23}$$

$$\frac{\partial \tilde{\tau}_1}{\partial \bar{z}} + \frac{\partial \tilde{\sigma}_\theta^{(1)}}{\partial \tilde{\theta}} + (\tilde{\sigma}_\theta^{(0)} - \tilde{\sigma}_\Phi^{(0)}) \cot \theta_0 = 0 \tag{24}$$

$$\frac{\partial \tilde{u}_i}{\partial \bar{z}} = 0 \qquad (i = 0, 1) \tag{25}$$

$$\tilde{u}_0 + \frac{\partial \tilde{v}_0}{\partial \tilde{\theta}} = \tilde{\sigma}_\theta^{(0)} - \nu\tilde{\sigma}_\Phi^{(0)} + \Theta(\theta_0, \bar{z}) \tag{26}$$

$$\tilde{u}_1 + \frac{\partial \tilde{v}_1}{\partial \tilde{\theta}} = \tilde{\sigma}_\theta^{(1)} - \nu\tilde{\sigma}_\Phi^{(1)} + \tilde{\theta}\frac{\partial \Theta}{\partial \theta}(\theta_0, \bar{z}) \tag{27}$$

$$\tilde{u}_0 = \tilde{\sigma}_\Phi^{(0)} - \nu\tilde{\sigma}_\theta^{(0)} + \Theta(\theta_0, \bar{z}) \tag{28}$$

$$\tilde{u}_1 + \tilde{v}_0 \cot \theta_0 = \tilde{\sigma}_\Phi^{(1)} - \nu\tilde{\sigma}_\theta^{(1)} + \tilde{\theta}\frac{\partial \Theta}{\partial \theta}(\theta_0, \bar{z}) \tag{29}$$

$$\frac{\partial \tilde{v}_i}{\partial \bar{z}} + \frac{\partial \tilde{u}_i}{\partial \tilde{\theta}} = 0 \qquad (i = 0, 1). \tag{30}$$

The solution of these equations must also satisfy the stress-free boundary condition on the inner and outer surfaces that

$$\tilde{\sigma}_r^{(0)} = \tilde{\sigma}_r^{(1)} = \cdots = 0 \qquad \tilde{\tau}_0 = \tilde{\tau}_1 = \cdots = 0 \qquad (\bar{z} = 0, 1)$$

and an attempt will be made to satisfy the conditions on the outer edge ($\theta = \theta_0$) that either the surface is stress-free, so that

$$\tilde{\sigma}_\theta^{(0)} = \tilde{\sigma}_\theta^{(1)} = \cdots = 0 \qquad \tilde{\tau}_0 = \tilde{\tau}_1 = \cdots = 0 \qquad (\tilde{\theta} = 0)$$

or the surface is clamped so that

$$\tilde{u}_0 = \tilde{u}_1 = \cdots = 0 \qquad \tilde{v}_0 = \tilde{v}_1 = \cdots = 0 \qquad (\tilde{\theta} = 0).$$

SOLUTIONS OF THE THIN-SHELL BOUNDARY LAYER EQUATIONS

The solution to the equations derived above for the "thin-shell" boundary layer is obtained by first observing that the $\bar{z}$-dependancy of the displacements, and hence of the in-plane stresses, can be determined immediately from the requirements eqns (25 and (30) that the radial normal strain and the shear strain components vanish. With the $\bar{z}$-dependancy of the in-plane stresses now known and expressed in terms of the displacement components, the radial equation of equilibrium eqn (21) or (22) can be integrated with respect to $\bar{z}$ to yield expressions for the shear stress in terms of the displacement components and the $\bar{z}$-coordinate explicitly. The stress-free boundary condition on the inner surface serves to fix the resultant function of integration, while the boundary condition on the outer surface yields one of the equations governing the displacement components. With the $\bar{z}$-dependancy of the shear stress now known, the meridional equation of equilibrium eqn (23) or (24) can be integrated with respect to $\bar{z}$ to yield expressions for the radial normal stress in terms of the displacement components and the $\bar{z}$-coordinate explicitly. The stress-free boundary condition on the inner surface again serves to fix the resulting function of integration, while the boundary condition on the outer surface yeilds the additional equation required to determine the displacement components. In both the first order and the second order solutions, the equations governing the displacement components can be reduced to an equation analogous to the equation governing a beam on an elastic foundation. This equation is also known to be characteristic of the equations governing a heated, thin shell. In the paragraphs to follow, the details of the solutions will be presented for both the first order and the second order corrections.

As noted above, the $\bar{z}$-dependancy of the first order displacements can be explicitly defined in terms of the functions $\tilde{U}_0$, $\tilde{V}_0$, where

$$\tilde{u}_0 = \tilde{U}_0(\tilde{\theta}) \qquad \tilde{v}_0 = \tilde{V}_0(\tilde{\theta}) - \bar{z}\tilde{U}_0' \qquad (\ )' = \frac{d(\ )}{d\tilde{\theta}}.$$

The in-plane stress components then become

$$(1-\nu^2)(\tilde{\sigma}_\theta^{(0)}, \tilde{\sigma}_\phi^{(0)}) = (1+\nu)[\tilde{U}_0 - \Theta(\theta_0, \bar{z})] + (1, \nu)(\tilde{V}_0' - \bar{z}\tilde{U}_0''). \tag{31}$$

With the $\bar{z}$-dependancy of the in-plane stresses determined explicitly, it follows from the meridional equation of equilibrium eqn (23) and the boundary condition on $\bar{z} = 0$ that

$$(1-\nu^2)\tilde{\tau}_0 = -\bar{z}[(1+\nu)\tilde{U}_0' + \tilde{V}_0''] + \bar{z}^2\tilde{U}_0'''/2. \tag{32}$$

Further, with the $\bar{z}$-dependancy of the shear stress determined explictly, it follows from the radial equation of equilibrium eqn (21) and the boundary condition at $\bar{z} = 0$ that

$$(1-\nu^2)\tilde{\sigma}_r^{(0)} = -2(1+\nu)\Xi(\theta_0, \bar{z}) + (1+\nu)\bar{z}(2\tilde{U}_0 + \tilde{V}_0') + \bar{z}^2\tilde{V}_0'''/2 - \bar{z}^3\tilde{U}_0^{IV}/6. \tag{33}$$

The equations relating the displacement components $\tilde{U}_0$, $\tilde{V}_0$ then follow from invoking the stress-free boundary conditions on $\tilde{\tau}_0, \tilde{\sigma}_r^{(0)}$ at $\bar{z} = 1$. Thus, with

$$[\tilde{U}_0''/2 - (1+\nu)\tilde{U}_0 - \tilde{V}_0']' = O$$

$$-\tilde{U}_0^{IV}/6 + \tilde{V}_0'''/2 + (1+\nu)(2\tilde{U}_0 + \tilde{V}_0') = 2(1+\nu)\Theta_m(\theta_0)$$

it follows that

$$\tilde{V}_0' = \tilde{U}_0''/2 - (1+\nu)\tilde{U}_0 + B_0 \tag{34}$$

where $\tilde{U}_0$ is the solution of

$$\tilde{U}_0^{IV} + 4m^4\tilde{U}_0 = 12(1+\nu)[2\Theta_m(\theta_0) - B_0] \tag{35}$$

and $m^4 = 3(1-\nu^2)$.

The solution for $\tilde{U}_0$ that is well behaved as $\tilde{\theta} \to -\infty$ can be taken in the form

$$\tilde{U}_0 = [2\Theta_m(\theta_0) - B_0]/(1-\nu) + e^{m\tilde{\theta}}(C_0 \cos m\tilde{\theta} + D_0 \sin m\tilde{\theta}). \tag{36}$$

With $\tilde{U}_0(\tilde{\theta})$ defined, the remaining displacement component $\tilde{V}_0(\tilde{\theta})$ is obtained on integrating eqn

(34) and leads to

$$\tilde{V}_0 = E_0 + 2\tilde{\theta}[B_0 - (1+\nu)\Theta_m(\theta_0)]/(1-\nu) + \frac{m}{2}e^{m\tilde{\theta}}\left\{\left[C_0\left(1-\frac{1+\nu}{m^2}\right) + D_0\left(1+\frac{1+\nu}{m^2}\right)\right]\cos m\tilde{\theta} + \left[D_0\left(1-\frac{1+\nu}{m^2}\right) - C_0\left(1+\frac{1+\nu}{m^2}\right)\right]\sin m\tilde{\theta}\right\}. \tag{37}$$

The constant B_0 is determined by the requirement that the form of the radial displacement obtained here match that obtained above in the interior region. The remaining constants C_0, D_0, E_0 are obtained from the conditions at the edge $\theta = \theta_0$. For convenience, the details of this calculation are deferred until the second order solution is obtained.

Proceeding in a similar fashion for the second order solution, the $\bar{z}$-dependancy of the displacements can again be explicitly defined in terms of functions $\tilde{U}_1$, $\tilde{V}_1$, where

$$\tilde{u}_1 = \tilde{U}_1(\tilde{\theta}) \qquad \tilde{v}_1 = \tilde{V}_1(\tilde{\theta}) - \bar{z}\tilde{U}_1'(\tilde{\theta}).$$

The in-plane stress components then become

$$(1-\nu^2)(\tilde{\sigma}_\theta^{(1)}, \tilde{\sigma}_\phi^{(1)}) = (1+\nu)\left[\tilde{U}_1 - \tilde{\theta}\frac{\partial\Theta}{\partial\theta}(\theta_0, \bar{z})\right] + (1, \nu)(\tilde{V}_1' - \bar{z}\tilde{U}_1'') + (\nu, 1)(\tilde{V}_0 - \bar{z}\tilde{U}_0')\cot\theta_0. \tag{38}$$

Proceeding as above with the equations of equilibrium eqns (22) and (24) we also obtain

$$(1-\nu^2)\tilde{\tau}_1(\tilde{\theta}, \bar{z}) = (1+\nu)\frac{\partial\Xi}{\partial\theta}(\theta_0, \bar{z}) - (1+\nu)\bar{z}\tilde{U}_1' - \bar{z}(\tilde{V}_1 - \bar{z}\tilde{U}_1'/2)'' - \bar{z}(\tilde{V}_0 - \bar{z}\tilde{U}_0'/2)'\cot\theta_0 \tag{39}$$

$$(1-\nu^2)\tilde{\sigma}_r^{(1)}(\tilde{\theta}, \bar{z}) = 2(1+\nu)\left[\bar{z}\tilde{U}_1 - \tilde{\theta}\frac{\partial\Xi}{\partial\theta}(\theta_0, \bar{z})\right] + \bar{z}[\bar{z}\tilde{V}_1'''/2 + (1+\nu)\tilde{V}_1' - \bar{z}^2\tilde{U}_1^{\mathrm{IV}}/6] + \bar{z}[(1+\nu)(\tilde{V}_0 - \bar{z}\tilde{U}_0') + \bar{z}(3-2\bar{z})\tilde{U}_0'''/6]\cot\theta_0. \tag{40}$$

The equations relating the displacement components $\tilde{U}_1$, $\tilde{V}_1$ then follow from invoking the stress-free boundary conditions on $\tilde{\tau}_1, \tilde{\sigma}_r^{(1)}$ at $\bar{z} = 1$. Thus, with

$$[(1+\nu)\tilde{U}_1 + (\tilde{V}_1 - \tilde{U}_1'/2)' + (\tilde{V}_0 - \tilde{U}_0'/2)\cot\theta_0]' = (1+\nu)\frac{\mathrm{d}\Theta_m}{\mathrm{d}\theta}(\theta_0)$$

$$\tilde{V}_1'''/2 + (1+\nu)\tilde{V}_1' - \tilde{U}_1^{\mathrm{IV}}/6 + 2(1+\nu)\left[\tilde{U}_1 - \tilde{\theta}\frac{\mathrm{d}\Theta_m}{\mathrm{d}\theta}(\theta_0)\right] + [(1+\nu)(\tilde{V}_0 - \tilde{U}_0') - \tilde{U}_0'''/6]\cot\theta_0 = 0$$

it follows that

$$\tilde{V}_1' = \tilde{U}_1''/2 - (1+\nu)\tilde{U}_1 - (\tilde{V}_0 - \tilde{U}_0'/2)\cot\theta_0 + (1+\nu)\tilde{\theta}\frac{\mathrm{d}\Theta_m}{\mathrm{d}\theta}(\theta_0) + B_1 \tag{41}$$

where $\tilde{U}_1$ is the solution of

$$\tilde{U}_1^{\mathrm{IV}} + 4m^4\tilde{U}_1 = 4m^4\tilde{\theta}\frac{\mathrm{d}\Theta_m}{\mathrm{d}\theta}(\theta_0) - 12(1+\nu)B_1 - 2\tilde{U}_0'''\cot\theta_0. \tag{42}$$

The solution for $\tilde{U}_1(\tilde{\theta})$ that is well behaved as $\tilde{\theta} \to -\infty$ can be taken in the form

$$\tilde{U}_1 = -B_1/(1-\nu) + \tilde{\theta}\frac{d\Theta_m}{d\theta}(\theta_0) + e^{m\tilde{\theta}}\Big[C_1 \cos m\tilde{\theta} + D_1 \sin m\tilde{\theta} - \frac{\tilde{\theta}}{2}(D_0 \sin m\tilde{\theta} + C_0 \cos m\tilde{\theta}) \cot \theta_0\Big]. \quad (43)$$

With $\tilde{U}_1(\tilde{\theta})$ defined, the remaining displacement component $\tilde{V}_1(\tilde{\theta})$ is obtained on integrating eqn (41) and leads to

$$\tilde{V}_1 = \tilde{U}_1'(\tilde{\theta})/2 + E_1 + \tilde{\theta}\Big(\frac{2B_1}{1-\nu} - E_0 \cot \theta_0\Big) - \frac{\tilde{\theta}^2}{1-\nu}[B_0 - (1+\nu)\Theta_m(\theta_0)] \cot \theta_0 + \frac{1+\nu}{4m^2} e^{m\tilde{\theta}}\{(C_0 \sin m\tilde{\theta} - D_0 \cos m\tilde{\theta}) \cot \theta_0 + m\tilde{\theta}[(C_0 + D_0) \sin m\tilde{\theta} + (C_0 - D_0) \cos m\tilde{\theta}] \cot \theta_0 - 2m[(C_1 - D_1) \cos m\tilde{\theta} + (C_1 - D_1) \sin m\tilde{\theta}]\}. \quad (44)$$

The constant B_1 is determined in terms of A_0 by the requirement that the form of the radial displacement determined here match that determined above for the interior region. The constant A_0 is determined in terms of E_0 by a similar matching requirement involving the meridional displacement components. Finally, the constant E_0 and the remaining constants C_1, D_1, E_1 are obtained from conditions at the edge $\theta = \theta_0$. However, before discussing the matching conditions between this "thin-shell" boundary layer solution and the interior solution, let us look at the implications of the boundary conditions at the edge $\theta = \theta_0$.

It is apparent from an examination of the forms obtained above for the displacement componests that the clamped edge conditions can be satisfied at $\theta = \theta_0$ for $\bar{z}$-arb. by requiring that

$$\tilde{U}_0(0),\ \tilde{V}_0(0),\ \tilde{U}_0'(0),\ \tilde{U}_1(0),\ \tilde{V}_1(0),\ \tilde{U}_1'(0) = 0.$$

The first three conditions are sufficient to determine the three constants C_0, D_0, E_0 assuming that B_0 has been determined by the matching conditions. The remaining three conditions are sufficient to determine the three constants C_1, D_1, E_1 assuming that B_1 has been determined by the matching conditions. Thus, the clamped conditions can be readily satisfied leading to a solution requiring a boundary layer of meridional length of the order of $(ah)^{1/2}$ in order to adjust the solution in the interior to the edge boundary conditions.

Alternatively, it is apparent from an examination of the form of the meridional stress components eqns (31) and (38) that no choice of constants of integration could satisfy the stress-free boundary conditions for an arbitrary temperature distribution. Further, though the stress component $\bar{\tau}_0$ can be made to vanish at $\tilde{\theta} = 0$ by a suitable choice of C_0, D_0, the edge value of the stress component $\bar{\tau}_1$ depends on the temperature distribution and hence cannot be made to vanish for $\bar{z}$-arb. Thus, the forms of the stress components obtained above as being valid in the "thin-shell" boundary layer are incapable of satisfying the stress-free boundary conditions. It is therefore necessary, as noted before by Cole[1], to introduce an additional boundary layer, located between the "thin-shell" boundary layer and the edge, in order to adjust the solution in the "thin-shell" layer to the stress-free boundary conditions. The details of this "Elasticity" boundary layer solution will be presented in subsequent sections. However, before we address this problem, it is necessary to complete the formulation of the "thin-shell" boundary layer solution by presenting the details of the matching requirements which establish B_0 and the relationship of B_1 to A_0.

MATCHING CONDITIONS BETWEEN THE INTERIOR AND THIN-SHELL LAYER SOLUTIONS

The essential feature of the method of matched asymptotic expansions is that the solution in adjacent regions has the same asymptotic expansion in a region intermediate to the two regions when the two solutions are expressed in a coordinate appropriate to that intermediate region. In constructing the solution forms in the intermediate region, we define a coordinate θ^* and a function $\eta(\bar{h})$ where

$$\theta^* = (\theta - \theta_0)/\eta(\bar{h})$$

and $\eta(\bar{h})$ is such that

$$\mathscr{L}_{\bar{h}\to 0, \theta^*\text{-fixed}} [\eta(\bar{h})/\bar{h}^{1/2}, \eta(\bar{h})] = (\infty, 0).$$

The definition of $\eta(\bar{h})$ implies that we are examining the solution in the interior region near $\theta = \theta_0$, and the solution in the "thin-shell" boundary for large values of $|\tilde{\theta}|$. In the paragraphs which follow, we present the details of this matching process for the radial displacement ($\bar{u}$), and present only the results of this process for the remaining stress and displacement variables as the mechanics of the process is quite similar in all cases.

The matching requirement for the radial displacement requires that

$$\mathscr{L}_{\bar{h}\to 0, \theta^*\text{-fixed}} \{[\bar{u}_0(\theta, \bar{z}) - \tilde{u}_0(\tilde{\theta}, \bar{z})] + \bar{h}^{1/2}[\bar{u}_1(\theta, \bar{z}) - \tilde{u}_1(\tilde{\theta}, \bar{z})] + 0(\bar{h})\} = 0 \tag{45}$$

to all orders of magnitude of $\eta(\bar{h})$. With the components $\bar{u}_0$, $\tilde{u}_0$, etc. of the displacement given in eqns (19), (20), (36), (43), the required expansions in the intermediate region are obtained by expressing the solution in the interior region near $\theta = \theta_0$ by

$$\bar{u}_0 \approx \Theta_m(\theta_0) + \eta(\bar{h})\theta^* \frac{d\Theta_m}{d\theta}(\theta_0) + 0(\eta^2)$$

$$\bar{u}_1 \approx -A_0 \cos\theta_0 + 0(\eta)$$

and the limiting form of the "thin-shell" boundary layer solution as

$$\tilde{u}_0 \approx [2\Theta_m(\theta_0) - B_0]/(1-\nu) + \text{T.S.T.}$$

$$\tilde{u}_1 \approx -B_1/(1-\nu) + \frac{\eta\theta^*}{\bar{h}^{1/2}} \frac{d\Theta_m}{d\theta}(\theta_0) + \text{T.S.T.}$$

where T.S.T. expresses the fact that the remaining terms are transcendentally small, as $\bar{h} \to 0$.

For first order matching, we require simply that the leading terms in eqn (45) match with the result that

$$B_0 = (1+\nu)\Theta_m(\theta_0).$$

This has the important consequence that the term proportional to $\tilde{\theta}$ in $\tilde{V}_0(\tilde{\theta})$ (see eqn 37) and the term proportional to $\tilde{\theta}^2$ in $\tilde{V}_1(\tilde{\theta})$ (see eqn 44) vanish identically.

For second order matching, we divide the matching requirement eqn (45) by $\bar{h}^{1/2}$ and obtain the result that

$$B_1 = (1-\nu)A_0 \cos\theta_0$$

provided that

$$\mathscr{L}_{\bar{h}\to 0} (\eta^2/\bar{h}^{1/2}) = 0.$$

Note also that the terms proportional to θ^* cancel.

On applying the matching requirement to the meridional displacement ($\bar{v}$), we obtain the result that

$$E_0 = A_0 \sin\theta_0$$

from first order matching, and a requirement on the behavior of $\bar{v}_1(\theta, \bar{z})$ near $\theta = \theta_0$ provided that B_1 is given by the condition noted above. Further details on this condition will not be presented as this requires the development of the third order solution in the interior which is beyond our current needs. Suffice it to say that this latter condition fixes the amplitude of a rigid body component of the second order solution in terms of E_1 and the temperature distribution at the edge.

On applying the matching requirement to the stress components, we obtain no new information but do obtain a verification of results obtained above. In particular, if we apply the matching requirement to the meridional normal stress component ($\bar{\sigma}_\theta$), we verify that the limiting form of $\tilde{\sigma}_\theta^{(0)}(\tilde{\theta}, \bar{z})$ agrees with $\bar{\sigma}_\theta^{(0)}(\theta_0, \bar{z})$ from the first order matching condition, and obtain the requirement that

$$\mathop{\mathscr{L}}_{\bar{h}\to 0,\theta^*\text{-fixed}} [B_1 - (1-\nu)\tilde{V}_0 \cot\theta_0] = 0$$

from the second order matching condition. This latter requirement is, of course, consistent with the results obtained above for E_0, B_1. Similar results are also obtained when the matching requirement is applied to the circumferential normal stress component ($\bar{\sigma}_\phi$).

On applying the matching requirement to the shear stress component ($\bar{\tau}$), we note that first order matching is achieved as $\tilde{\tau}_0(\tilde{\theta}, \bar{z})$ is transcendentally small in the intermediate region. Proceeding with second order matching, we verify that the limiting form of $\tilde{\tau}_1(\tilde{\theta}, \bar{z})$ agrees with $\bar{\tau}_0(\theta_0, \bar{z})$.

Finally, on applying the matching requirement to the radial normal stress component ($\bar{\sigma}_r$), we verify that the limiting form of $\tilde{\sigma}_r^{(0)}(\tilde{\theta}, \bar{z})$ agrees with $\bar{\sigma}^{(0)}(\theta_0, \bar{z})$ from the first order matching condition, and note that the second order matching condition is satisfied identically.

Thus, the matching conditions establish the constants B_0, B_1, A_0 so that the interior solution and the "thin-shell" boundary layer solution match asymptotically in the intermediate region of overlap. For the clamped boundary condition, this completes the formulation of the solution as the remaining constants of integration are all determined by conditions at the edge $\theta = \theta_0$. For the stress-free boundary condition, however, we must proceed to find the form of the solution in the "Elasticity" layer in order to determine what conditions must be satisfied by the "thin-shell" solution at the edge $\tilde{\theta} = 0$. In the sections to follow, this "Elasticity" layer solution is developed and used with the matching principle to determine the required additional conditions.

ELASTICITY BOUNDARY LAYER EQUATIONS

In order to satisfy the stress-free boundary condition along $\theta = \theta_0$, it is necessary to impose a boundary layer within the "thin-shell" boundary layer. As the meridional extent of this "Elasticity" layer is expected to be of order h, we introduce the coordinate $\hat{\theta}$, where

$$\theta = \theta_0 + \bar{h}\hat{\theta}$$

and assume that the solution in this region has the following form as $\bar{h} \to 0$

$$\begin{aligned}
\bar{u}(\hat{\theta}, \bar{z}, \bar{h}) &= \hat{u}_0(\hat{\theta}, \bar{z}) + \bar{h}^{1/2}\hat{u}_1(\hat{\theta}, \bar{z}) + \bar{h}\hat{u}_2(\hat{\theta}, \bar{z}) + \bar{h}^{3/2}\hat{u}_3(\hat{\theta}, \bar{z}) + 0(\bar{h}^2)\\
\bar{v}(\hat{\theta}, \bar{z}, \bar{h}) &= \bar{h}^{1/2}[\hat{v}_0(\hat{\theta}, \bar{z}) + \bar{h}^{1/2}\hat{v}_1(\hat{\theta}, \bar{z}) + \bar{h}\hat{v}_2(\hat{\theta}, \bar{z}) + 0(\bar{h}^{3/2})]\\
\bar{\sigma}_r(\hat{\theta}, \bar{z}, \bar{h}) &= \hat{\sigma}_r^{(0)}(\hat{\theta}, \bar{z}) + \bar{h}^{1/2}\hat{\sigma}_r^{(1)}(\hat{\theta}, \bar{z}) + \bar{h}\hat{\sigma}_r^{(2)}(\hat{\theta}, \bar{z}) + 0(\bar{h}^{3/2})\\
\bar{\tau}(\hat{\theta}, \bar{z}, \bar{h}) &= \hat{\tau}_0(\hat{\theta}, \bar{z}) + \bar{h}^{1/2}\hat{\tau}_1(\hat{\theta}, \bar{z}) + \bar{h}\hat{\tau}_2(\hat{\theta}, \bar{z}) + 0(\bar{h}^{3/2})\\
\bar{\sigma}_\theta(\hat{\theta}, \bar{z}, \bar{h}) &= \hat{\sigma}_\theta^{(0)}(\hat{\theta}, \bar{z}) + \bar{h}^{1/2}\hat{\sigma}_\theta^{(1)}(\hat{\theta}, \bar{z}) + \bar{h}\hat{\sigma}_\theta^{(2)}(\hat{\theta}, \bar{z}) + 0(\bar{h}^{3/2})\\
\bar{\sigma}_\phi(\hat{\theta}, \bar{z}, \bar{h}) &= \hat{\sigma}_\phi^{(0)}(\hat{\theta}, \bar{z}) + \bar{h}^{1/2}\hat{\sigma}_\phi^{(1)}(\hat{\theta}, \bar{z}) + \bar{h}\hat{\sigma}_\phi^{(2)}(\hat{\theta}, \bar{z}) + 0(\bar{h}^{3/2}).
\end{aligned}$$

When this form is substituted into the governing eqns (1)–(6), and repeated limits $\bar{h} \to 0$ are taken, we find that the following hierarchy of equations govern the solution

$$\frac{\partial \hat{\sigma}_r^{(i)}}{\partial \bar{z}} + \frac{\partial \hat{\tau}_i}{\partial \hat{\theta}} = 0 \qquad (i = 0, 1) \tag{45}$$

$$\frac{\partial \hat{\tau}_i}{\partial \bar{z}} + \frac{\partial \hat{\sigma}_\theta^{(i)}}{\partial \hat{\theta}} = 0 \qquad (i = 0, 1) \tag{46}$$

$$\frac{\partial \hat{u}_i}{\partial \bar{z}} = 0 \qquad (i = 0, 1) \tag{47}$$

$$\frac{\partial \hat{u}_2}{\partial \bar{z}} = \hat{\sigma}_r^{(0)} - \nu(\hat{\sigma}_\theta^{(0)} + \hat{\sigma}_\phi^{(0)}) + \Theta(\theta_0, \bar{z}) \tag{48}$$

$$\frac{\partial \hat{u}_3}{\partial \bar{z}} = \hat{\sigma}_r^{(1)} - \nu(\hat{\sigma}_\theta^{(1)} + \hat{\sigma}_\phi^{(1)}) \tag{49}$$

$$\frac{\partial \hat{v}_0}{\partial \hat{\theta}} = 0 \tag{50}$$

$$\frac{\partial \hat{v}_1}{\partial \hat{\theta}} + \hat{u}_0 = \hat{\sigma}_\theta^{(0)} - \nu(\hat{\sigma}_r^{(0)} + \hat{\sigma}_\phi^{(0)}) + \Theta(\theta_0, \bar{z}) \tag{51}$$

$$\frac{\partial \hat{v}_2}{\partial \hat{\theta}} + \hat{u}_1 = \hat{\sigma}_\theta^{(1)} - \nu(\hat{\sigma}_r^{(1)} + \hat{\sigma}_\phi^{(1)}) \tag{52}$$

$$\hat{u}_0 = \hat{\sigma}_\phi^{(0)} - \nu(\hat{\sigma}_\theta^{(0)} + \hat{\sigma}_r^{(0)}) + \Theta(\theta_0, \bar{z}) \tag{53}$$

$$\hat{u}_1 + \hat{v}_0 \cot \theta_0 = \hat{\sigma}_\phi^{(1)} - \nu(\hat{\sigma}_\theta^{(1)} + \hat{\sigma}_r^{(1)}) \tag{54}$$

$$\frac{\partial \hat{u}_0}{\partial \hat{\theta}} = 0 \tag{55}$$

$$\frac{\partial \hat{v}_0}{\partial \bar{z}} + \frac{\partial \hat{u}_1}{\partial \hat{\theta}} = 0 \tag{56}$$

$$\frac{\partial \hat{v}_1}{\partial \bar{z}} + \frac{\partial \hat{u}_2}{\partial \hat{\theta}} = 2(1+\nu)\hat{\tau}_0 \tag{57}$$

$$\frac{\partial \hat{v}_2}{\partial \bar{z}} + \frac{\partial \hat{u}_3}{\partial \hat{\theta}} - \bar{z}\frac{\partial \hat{u}_1}{\partial \hat{\theta}} - \hat{v}_0 = 2(1+\nu)\hat{\tau}_1. \tag{58}$$

The solution of these equations must also satisfy the stress-free conditions that

$$\hat{\sigma}_r^{(0)}, \hat{\sigma}_r^{(1)}, \ldots = 0 \qquad \hat{\tau}_0, \hat{\tau}_1, \ldots = 0 \qquad (\bar{z} = 0, 1)$$

on the inner and outer surfaces, and

$$\hat{\sigma}_\theta^{(0)}, \hat{\sigma}_\theta^{(1)}, \ldots = 0 \qquad \hat{\tau}_0, \hat{\tau}_1, \ldots = 0 \qquad (\hat{\theta} = 0)$$

on the edge $\theta = \theta_0$.

In addition to the boundary conditions, the solution of these equations must match the solution of the thin-shell boundary layer equations in some sense as $\bar{h} \to 0$. This matching procedure will serve to identify not only the constants of integration of the "thin-shell" solution, but will also lead to some physical interpretation of the matching conditions themselves. The details of this procedure will be deferred to a later section while attention is devoted to obtaining the form of the solution itself.

SOLUTIONS OF THE ELASTICITY LAYER EQUATIONS

The solution of the boundary layer equations can most conveniently be formulated in terms of the stresses $\hat{\sigma}_r^{(i)}$, $\hat{\sigma}_\theta^{(i)}$, $\hat{\tau}_i$ $(i = 0, 1)$ to form an equivalent plane strain problem. As a preliminary, however, the displacements accompanying these stresses must first be determined, and compatibility equations derived from the equations relating the higher order displacement components to augment the existing equilibrium eqns (45) and (46).

The displacements accompanying the first and second order stresses follow immediately from eqns (47), (50), (55) and (56) which essentially require that the radial normal, meridional normal and shear strain components vanish identically. Thus, with

$$\hat{u}_0 = \hat{A}_0 = \text{const.} \qquad v_0 = \hat{V}_0(\bar{z})$$

$$\hat{u}_1 = \hat{U}_1(\hat{\theta}) \qquad \frac{\mathrm{d}\hat{V}_0}{\mathrm{d}\bar{z}} = -\frac{\mathrm{d}\hat{U}_1}{\mathrm{d}\hat{\theta}}$$

it follows that

$$\frac{\mathrm{d}\hat{U}_1}{\mathrm{d}\hat{\theta}} = -\frac{\mathrm{d}\hat{V}_0}{\mathrm{d}\bar{z}} = A_2 = \text{constant,}$$

and

$$\hat{u}_1 = \hat{A}_1 + \hat{A}_2\hat{\theta} \qquad \hat{v} = \hat{B}_1 - \hat{A}_2\bar{z} \qquad (\hat{A}_1, \hat{B}_1 = \text{const.}).$$

Also, we obtain

$$\hat{\sigma}_\phi^{(0)} = \hat{A}_0 - \Theta(\theta_0, \bar{z}) + \nu(\hat{\sigma}_\theta^{(0)} + \hat{\sigma}_r^{(0)}) \tag{59}$$

$$\hat{\sigma}_\phi^{(1)} = \hat{A}_1 + \hat{A}_2\hat{\theta} + (\hat{B}_1 - \hat{A}_2\bar{z}) \cot\theta_0 + \nu(\hat{\sigma}_\theta^{(1)} + \hat{\sigma}_r^{(1)}). \tag{60}$$

The compatibility equation for the first order stresses is obtained by eliminating the displacement components $\hat{u}_2$, $\hat{v}_1$ from eqns (48), (51) and (57). Thus, on taking the appropriate derivatives, there results

$$\frac{\partial^2}{\partial\bar{z}^2}[\hat{\sigma}_\theta^{(0)} - \nu(\hat{\sigma}_r^{(0)} + \hat{\sigma}_\phi^{(0)})] + \frac{\partial^2}{\partial\hat{\theta}^2}[\hat{\sigma}_r^{(0)} - \nu(\hat{\sigma}_\theta^{(0)} + \hat{\sigma}_\phi^{(0)})] = 2(1+\nu)\frac{\partial^2\hat{\tau}_0}{\partial\bar{z}\partial\hat{\theta}} - \frac{\partial^2\Theta}{\partial\bar{z}^2}(\theta_0, \bar{z}).$$

It should be noted that this equation could have alternatively been derived directly from the compatibility equations given by Lur'ye[9]. In particular it is the first order form of the equation ($Q_{33} = 0$). Finally, on eliminating $\hat{\sigma}_\phi^{(0)}$ by means of eqn (59), we obtain the following equation to augment eqns (45) and (46) for the determination of the stress components

$$\frac{\partial^2\hat{\sigma}_r^{(0)}}{\partial\hat{\theta}^2} + \frac{\partial^2\hat{\sigma}_\theta^{(0)}}{\partial\bar{z}^2} - \nu\nabla^2(\hat{\sigma}_r^{(0)} + \hat{\sigma}_\theta^{(0)}) = 2\frac{\partial^2\hat{\tau}_0}{\partial\hat{\theta}\partial\bar{z}} - \frac{\partial^2\Theta}{\partial\bar{z}^2}(\theta_0, \bar{z}) \tag{61}$$

where

$$\nabla^2(\;) = \left(\frac{\partial^2}{\partial\bar{z}^2} + \frac{\partial^2}{\partial\hat{\theta}^2}\right)(\;).$$

As is well known, the solution of the equations of equilibrium eqns (45) and (46) that also satisfies eqn (61) can be expressed in terms of an Airy stress function $\phi^{(0)}$, where

$$\hat{\sigma}_\theta^{(0)} = \frac{\partial^2\phi^{(0)}}{\partial\bar{z}^2} \qquad \hat{\sigma}_r^{(0)} = \frac{\partial^2\phi^{(0)}}{\partial\hat{\theta}^2} \qquad \hat{\tau}_0 = \frac{\partial^2\phi^{(0)}}{\partial\bar{z}\partial\hat{\theta}}$$

and $\phi^{(0)}$ is the solution of

$$\nabla^4\phi^{(0)} = -\frac{1}{1-\nu}\cdot\frac{\partial^2\Theta}{\partial\bar{z}^2}(\theta_0, \bar{z}). \tag{62}$$

The stress-free boundary conditions can also be expressed in terms of $\phi^{(0)}$, and lead to the following equivalent representation

$$\phi^{(0)} = 0 \qquad \frac{\partial \phi^{(0)}}{\partial \hat{\theta}} = 0 \qquad \text{on} \qquad \hat{\theta} = 0, \qquad 0 \leq \bar{z} \leq 1$$

$$\phi^{(0)} = 0 \qquad \frac{\partial \phi^{(0)}}{\partial \bar{z}} = 0 \qquad \text{on} \qquad \bar{z} = 0, 1, \qquad -\infty < \hat{\theta} \leq 0. \tag{63}$$

Thus, the problem of determining the stresses in the "Elasticity" boundary layer is analogous to the problem of determining the transverse displacement of a laterally loaded, clamped rectangular plate. It should also be noticed that, following [3], this problem is identical to that of determining the stresses in the "Elasticity" layer associated with the thermal stress distribution in a thin circular disk. Much of the solution described in [3] will therefore be applicable to the present problem. Before proceeding with this solution, however, we return to complete the formulation of the equations governing the second order solution.

Proceeding as above, we obtain the compatibility equation for the second order stresses by eliminating the displacement componentts $\hat{u}_3$, $\hat{v}_2$ from eqns (49), (52) and (58). Thus, on taking the appropriate derivatives, there results

$$\frac{\partial^2}{\partial \bar{z}^2}[\hat{\sigma}_\theta^{(1)} - \nu(\hat{\sigma}_r^{(1)} + \hat{\sigma}_\phi^{(1)})] + \frac{\partial^2}{\partial \hat{\theta}^2}[\hat{\sigma}_r^{(1)} - \nu(\hat{\sigma}_\theta^{(1)} + \hat{\sigma}_\phi^{(1)})] = 2(1+\nu)\frac{\partial^2 \hat{\tau}_1}{\partial \hat{\theta}\partial \bar{z}}.$$

As above, it should be noted that this equation is simply the second order form of the compatibility equation ($Q_{33} = 0$) proposed by Lur'ye. Finally, on eliminating $\hat{\sigma}_\phi^{(1)}$ by means of eqn (60), we obtain the following equation to augment eqns (45) and (46) for the determination of the second order stress components

$$\frac{\partial^2 \hat{\sigma}_\theta^{(1)}}{\partial \bar{z}^2} + \frac{\partial^2 \hat{\sigma}_r^{(1)}}{\partial \hat{\theta}^2} - \nu \nabla^2(\hat{\sigma}_\theta^{(1)} + \hat{\sigma}_r^{(1)}) = 2\frac{\partial^2 \hat{\tau}_1}{\partial \hat{\theta}\partial \bar{z}}. \tag{64}$$

The solution of the equations of equilibrium eqns (45) and (46) that also satisfies eqn (64) can also be expressed in terms of an Airy stress function $\phi^{(1)}$, where

$$\hat{\sigma}_\theta^{(1)} = \frac{\partial^2 \phi^{(1)}}{\partial \bar{z}^2} \qquad \hat{\sigma}_r^{(1)} = \frac{\partial^2 \phi^{(1)}}{\partial \hat{\theta}^2} \qquad \hat{\tau}_1 = -\frac{\partial^2 \phi^{(1)}}{\partial \bar{z}\partial \hat{\theta}}$$

and $\phi^{(1)}$ is the solution of

$$\nabla^4 \phi^{(1)} = 0. \tag{65}$$

The stress-free boundary conditions can also be expressed in terms of $\phi^{(1)}$, and lead to equations analogous to eqns (63). In this case, however, the only solution of eqn (65) that satisfies homogeneous boundary conditions is the trivial solution. This, with

$$\hat{\sigma}_\theta^{(1)} = \hat{\sigma}_r^{(1)} = \hat{\tau}_1 = 0 \tag{66}$$

it follows that

$$\hat{\sigma}_\phi^{(1)} = (\hat{A}_1 + \hat{B}_1 \cot \theta_0) + \hat{A}_2(\hat{\theta} - \bar{z} \cot \theta_0). \tag{67}$$

Returning now to the problem of obtaining the solution of eqn (62), we note that for our present purposes it is necessary to obtain only the limiting form of that solution as $|\hat{\theta}| \to \infty$. It should be noted, however, that the complete solution of eqn (62) must be obtained numerically. An example solution is presented in [3] for the case of a temperature distribution such that $(\partial^2 \Theta/\partial \bar{z}^2)(\theta_0, \bar{z}) = \text{const}$. However, before we can apply the results of that example here, we must obtain the constants of integration $\hat{A}_i$ $(i = 0, 1, 2)$ which depend on matching this solution to that in the "thin-shell" layer. The details of this procedure are presented in the next section.

The required limiting form of the solution of eqn (62) is obtained, following [3] by assuming that the general solution is the superposition of the particular solution given by

$$-(1-\nu)\phi^{(0)}_{\text{part}} = \int_0^{\bar{z}} \Xi(\theta_0, x)\,dx + \bar{z}^2(1-\bar{z})\Theta_m(\theta_0) - \bar{z}^2(3-2\bar{z})\Xi_m(\theta_0)$$

and a correction, where the stresses due to the correction can be interpreted as due to a self-equilibrating meridional normal stress distributed over the edge $\hat{\theta} = 0$. If the stress distribution due to the correction can be shown to decay sufficiently as $|\hat{\theta}| \to \infty$, it is clear that the stresses associated with $\phi^{(0)}_{\text{part}}$ are the required limiting form. However, Knowles [10] has shown that this is indeed the case, and that the stresses due to a self-equilibrating system of loads applied to the finite end of a semi-infinite strip decay exponentially with distance from that end. Thus, applying Knowles' result here leads to the conclusion that

$$\hat{\sigma}_\theta^{(0)} \approx \frac{d\phi^{(0)}_{\text{part}}}{d\bar{z}^2} + \text{T.S.T.} \qquad \hat{\sigma}_r^{(0)}, \hat{\tau}_0 = \text{T.S.T.} \tag{68}$$

Having established the forms for the stresses and the radial displacement component of the second order solution, it is left only to determine the associated meridional displacement component to complete the solution. This is readily accomplished by integrating eqn (51) with respect to $\hat{\theta}$ noting the boundary conditions on $\phi^{(0)}$ and, following the results of the last paragraph, defining

$$\int_0^{\hat{\theta}} \hat{\sigma}_\theta^{(0)}(x, \bar{z})\,dx = \int_0^{\hat{\theta}} \frac{d^2\phi^{(0)}_{\text{part}}}{d\bar{z}^2}(\bar{z})\,dx + F_0(\hat{\theta}, \bar{z})/(1-\nu^2).$$

The function $F_0(\hat{\theta}, \bar{z})/(1-\nu^2)$ is the integral of that component of the meridional stress $\hat{\sigma}_\theta^{(0)}$ that decays exponentially with distance from the edge $\hat{\theta}=0$ and hence is bounded as $|\hat{\theta}| \to \infty$. Once the numerical solution of eqn (62) is obtained, the function $F_0(\hat{\theta},\bar{z})$ can readily be obtained by subtracting the particular solution for $\hat{\sigma}_\theta^{(0)}$ from the complete solution and integrating. Thus, assuming that $F_0(\hat{\theta}, \bar{z})$ is available, it follows from eqn (51) that

$$\begin{aligned}\hat{v}_1(\hat{\theta}, \bar{z}) = \hat{v}_1(0, \bar{z}) - \nu(1+\nu)\frac{\partial\phi^{(0)}}{\partial\hat{\theta}} + F_0(\hat{\theta}, \bar{z})\\ -(1+\nu)\hat{\theta}[\hat{A}_0 + 2(1-3\bar{z})\Theta_m(\theta_0) - 6(1-2\bar{z})\Xi_m(\theta_0)]\end{aligned} \tag{69}$$

where $\hat{v}_1(0, \bar{z})$ is as yet undetermined. The form of this function is determined in the next section from the matching principle.

MATCHING CONDITIONS BETWEEN THE THIN-SHELL AND ELASTICITY LAYER SOLUTIONS

As the solution in the "thin-shell" layer is unable to satisfy stress-free conditions at the edge $\tilde{\theta} = 0$, and the solution in the "Elasticity" layer has been constructed to satisfy those conditions, we must seek the conditions which ensure that the two forms of the solution match. In particular, as in a previous section, we seek the asymptotic expansion of the solution in each layer in a region intermediate to the two layers and the conditions under which these expansions agree.

In constructing the forms of the solution in this intermediate region, we define a coordinate θ^{**} and a function $\mu(\bar{h})$ where

$$\theta^{**} = (\theta - \theta_0)/\mu(\bar{h})$$

and $\mu(\bar{h})$ is such that

$$\mathop{\mathscr{L}}_{\bar{h},\theta^{**}\text{-fixed}} [\mu(\bar{h})/\bar{h}^{1/2}, \mu(\bar{h})/\bar{h}] = (0, \infty).$$

The definition of $\mu(\bar{h})$ implies that we are examining the form of the solution in the "thin-shell" layer in the region near $\tilde{\theta} = 0$, and the form of the solution in the "Elasticity" layer for large values of $|\hat{\theta}|$. In the paragraphs to follow, we present the details of this matching process for the meridional normal stress ($\bar{\sigma}_\theta$), and present only the results of this process for the remaining stress and displacement variables.

The matching requirement for the meridional normal stress requires that

$$\mathop{\mathscr{L}}_{\bar{h}\to 0,\theta^{**}\text{-fixed}} \{[\tilde{\sigma}_\theta^{(0)}(\tilde{\theta}, \bar{z}) - \hat{\sigma}_\theta^{(0)}(\hat{\theta}, \bar{z})] + \bar{h}^{1/2}[\tilde{\sigma}_\theta^{(1)}(\tilde{\theta}, \bar{z}) - \hat{\sigma}_\theta^{(1)}(\hat{\theta}, \bar{z})] + 0(\bar{h})\} = 0 \tag{70}$$

to all orders of magnitude of $\mu(\bar{h})$. With the components $\tilde{\sigma}_\theta^{(0)}$, $\tilde{\sigma}_\theta^{(1)}$ of the "thin-shell" solution given by eqns (31) and (38), the required expansion in the intermediate region can be constructed by expressing the solution near $\tilde{\theta}=0$ by

$$(1-\nu^2)\tilde{\sigma}_\theta^{(0)} \approx (1+\nu)[\tilde{U}_0(0) - \Theta(\theta_0, \bar{z})] + \tilde{V}_0'(0) - \bar{z}\tilde{U}_0''(0) + \tilde{\theta}[(1+\nu)\tilde{U}_0'(0) + \tilde{V}_0''(0) - \bar{z}\tilde{U}_0'''(0)] + 0(\tilde{\theta}^2)$$

$$(1-\nu^2)\tilde{\sigma}_\theta^{(1)} \approx (1+\nu)\tilde{U}_1(0) + \tilde{V}_1'(0) - \bar{z}\tilde{U}_1''(0) + \nu[\tilde{V}_0(0) - \bar{z}\tilde{U}_0'(0)]\cot\theta_0 + 0(\tilde{\theta})$$

where $\tilde{V}_0$, $\tilde{V}_1$ are defined by eqns (34) and (41). The limiting forms of the meridional stress in the "Elasticity" layer are given by eqns (66) and (68).

For first order matching, we require simply that the leading terms in eqn (70) match with the result that

$$\tilde{U}_0''(0) = -6(1+\nu)[\Theta_m(\theta_0) - 2\Xi_m(\theta_0)]. \tag{71}$$

For second order matching, we divide the matching requirement eqn (70) by $\bar{h}^{1/2}$ and obtain

$$\mathop{\mathscr{L}}_{\bar{h}\to 0,\theta^{**}\text{-fixed}} \Big\{\mu\theta^{**}\frac{1-2\bar{z}}{2\bar{h}}\tilde{U}_0'''(0) + [(1+\nu)\tilde{U}_1(0) + \tilde{V}_1'(0) + \nu\tilde{V}_0(0)\cot\theta_0] - \bar{z}[\tilde{U}_1''(0) + \nu\tilde{U}_0'(0)\cot\theta_0] + 0(\mu^2/\bar{h}^{3/2})\Big\} = 0.$$

Clearly, if this expression is to vanish for all $\bar{z}$, we must require that

$$\tilde{U}_0'''(0) = 0 \tag{72}$$

$$(1+\nu)\tilde{U}_1(0) + \tilde{V}_1'(0) + \nu\tilde{V}_0(0)\cot\theta_0 = 0 \tag{73}$$

$$\tilde{U}_1''(0) + \nu\tilde{U}_0'(0)\cot\theta_0 = 0 \tag{74}$$

provided that $\mu(\bar{h})$ is such that

$$\mathop{\mathscr{L}}_{\bar{h}\to 0} (\mu^2/\bar{h}^{3/2}) = 0.$$

To readers familiar with thin-shell theory, these equations have immediate physical significance in terms of the stress resultants familiar in shell theory. For convenience, however, we prefer to defer this discussion until all matching conditions are developed.

On applying the matching requirement to the remaining stress and displacement variables, we obtain no new information with which to fix the constants of the "thin-shell" solution save for a condition on the first order meridional displacement. Essentially, the remaining conditions serve to fix the constants of the "Elasticity" layer solution. This follows the observation that the edge value of the radial displacement is governed by the interior solution, while the edge value of the meridional displacement is governed by conditions on the "Elasticity" layer solution. In what follows, it is convenient to take $\bar{v}(\theta = \theta_0, \bar{z} = 1/2) = 0$ so that

$$\hat{B}_1 = \hat{A}_2/2$$

in order to more readily compare the present results with those of thin-shell theory.

If we apply the matching requirement to the circumferential normal stress component ($\bar{\sigma}_\phi$), we again obtain the result given by eqn (71) from the first order condition, while second order matching requires that

$$\hat{A}_1 + \hat{B}_1 \cot\theta_0 = \tilde{U}_1(0) + \tilde{V}_0(0)\cot\theta_0 \tag{75}$$

$$\hat{A}_2 = \tilde{U}_0'(0) \tag{76}$$

if we use eqns (73) and (74) to eliminate $\tilde{U}_1''(0)$, $\tilde{V}_1'(0)$.

If we apply the matching requirement to the radial displacement component ($\bar{u}$), we obtain the requirements that

$$\hat{A}_0 = \tilde{U}_0(0) \tag{77}$$

$$\hat{A}_1 = \tilde{U}_1(0) \tag{78}$$

from the first and second order matching conditions respectively.

If we apply the matching requirement to the meridional displacement component ($\bar{v}$), we obtain the condition that

$$\tilde{V}_0(0) = \hat{B}_1 \tag{79}$$

from the first order matching condition. As $\hat{B}_1$ has been determined above, this equation essentially fixes E_0 in terms of C_0, D_0. In order to carry out second order matching, we require the limiting form of $\hat{v}_1(\hat{\theta}, \bar{z})$ as $|\hat{\theta}| \to \infty$. This form is readily obtained from eqn (69) as $(\partial\phi^{(0)}/\partial\hat{\theta})$ becomes transcendentally small and $F_0(\hat{\theta}, \bar{z})$ becomes $F_0(\infty, \bar{z})$ by definition. Thus, it follows that second order matching is achieved provided that

$$\hat{v}_1(0, \bar{z}) = \tilde{V}_1(0) - \bar{z}\tilde{U}_1'(0) - F_0(\infty, \bar{z}) \tag{80}$$

and eqn (71) are satisfied. As we have already arbitrarily taken $\bar{v}(\theta = \theta_0, \bar{z} = 1/2) = 0$, we see that this implies that $\hat{v}_1(0, 1/2) = 0$, while eqn (80) can be interpreted as fixing E_1 in terms of C_0, C_1, D_0, D_1 and the temperature distribution.

When the matching requirement is applied to the shear stress component ($\bar{\tau}$), we find that the first order condition is satisfied identically as $\hat{\tau}_0(\hat{\theta}, \bar{z})$ is transcendentally small as $|\hat{\theta}| \to \infty$. Proceeding to the next order, we obtain the requirement that $\tilde{\tau}_0(0, \bar{z}) = 0$ which is equivalent to eqn (72). Thus, matching through two orders of magnitude has yielded no new information to accompany eqn (74) for fixing the constants of the second order "thin-shell" layer solution. It is necessary then to extend our study of matching requirements to order $\bar{h}$. However, as this requires developing the limiting form of $\hat{\tau}_2(\hat{\theta}, \bar{z})$ and the associated third order "Elasticity" layer solution that has otherwise not been required, we regret the interruption in the continuity of the presentation and proceed to present the details of the calculation in the Appendix.

With the limiting form of $\hat{\tau}_2(\hat{\theta}, \bar{z})$ given by eqn (A21) of the Appendix, and the expressions for the shear stress components $\tilde{\tau}_0$, $\tilde{\tau}_1$ given by eqns (32) and (39), it follows that matching to order $\bar{h}$ is achieved provided that

$$\tilde{U}_1'''(0) + \tilde{U}_0''(0)\cot\theta_0 = 6(1+\nu)\left[2\frac{d\Xi_m}{d\theta}(\theta_0) - \frac{d\Theta_m}{d\theta}(\theta_0)\right] \tag{81}$$

is satisfied and $\mu(\bar{h})$ is such that

$$\mathop{\mathscr{L}}_{\bar{h}\to 0}(\mu^2/\bar{h}^{3/2}) = 0.$$

Thus, by extending the order of the matching process, we have succeeded in obtaining an additional requirement for the determination of the constants of the second order "thin-shell" layer solution. Again, this condition has immediate physical significance in terms of the stress resultants of thin shell theory, and will be discussed in the next section.

As a final exercise in this section, we examine the matching requirement applied to the radial normal stress component ($\bar{\sigma}_r$). As with the shear stress component, we find that first order matching is achieved identically as $\hat{\sigma}_r^{(0)}(\hat{\theta}, \bar{z})$ is transcendentally small as $|\hat{\theta}| \to \infty$. However, unlike the case of the shear stress, the next term in the series for $\bar{\sigma}_r$ is $0(\bar{h})$ so that we must again look to the third order "Elasticity" layer solution in order to match the component $\bar{\sigma}_r^{(0)}$ of the "thin-shell" layer. Unfortunately, as above, the reader must refer to the Appendix to obtain a presentation of the details.

The matching condition to order $\bar{h}$ essentially requires that the limiting form of $\hat{\sigma}_r^{(2)}(\hat{\theta}, \bar{z})$ as given by eqn (A22) of the Appendix agree with the edge value of $\bar{\sigma}_r^{(0)}(\tilde{\theta}, \bar{z})$ as given by eqn (33). As can be shown by using the condition on $\tilde{U}_0''(0)$ obtained above, this matching condition is identically satisfied. Thus, all components of stress and displacement in the two layers are matched at least to within $0(\bar{h}^{1/2})$ compared to unity.

In summary, we have used the matching conditions to determine the constants of integration of the solution in the two layers. We have developed two equations (eqns 71 and 72) for the determination of C_0, D_0 of the first order solution, and two equations (eqns 74 and 81) for the determination of C_1, D_1 of the second order solution. With C_0, D_0 known, $\hat{A}_2$ (and hence $\hat{B}_1$) is determined from eqn (78). The constant E_0 (and hence A_0, B_1) then follows from eqn (79). At this stage, $\tilde{U}_0$, $\tilde{U}_1$ are completely defined so that $\hat{A}_0$, $\hat{A}_1$, $\hat{A}_2$ can be determined from eqns (76)–(78). Finally, assuming that the limiting form of $F_0(\hat{\theta}, \bar{z})$ is available, the constant E_1 is obtained from eqn (80). This completes the formulation of the solution. In the following section, we examine the implications of these results and note how they might be applied to practical problems.

SUMMARY AND DISCUSSION

In the previous sections, an approximate solution of the equations of Thermoelasticity has been obtained for an axisymmetric temperature distribution. The character of the solution is found to depend on the boundary condition at $\theta = \theta_0$. For the clamped condition, the solution is characterized by a "membrane" region away from the edge and a "thin-shell" boundary near the edge. These regions are themselves characterized by the solution varying significantly over meridional lengths comparable with the radius (a) and the intermediate length ($\sqrt{(ah)}$) respectively. For the stress-free boundary condition, the solution is similarly characterized except that there must be an "Elasticity" layer between the edge and the "thin-shell" layer. This layer is characterized by the stresses being all the same order of magnitude and varying significantly over a meridional length comparable with the thickness (h) of the shell.

We now proceed to show that the solution obtained above is identical to that obtained in [4] from the thin-shell equations. Clearly, this does not apply to the solution in the "Elasticity" layer and the implications of the solution in this region will be examined in a later paragraph.

The solution in the interior region is most readily compared as the notation for the displacements is essentially the same and the stress resultants are readily obtained from the stress components using the temperature integrals defined in the first section. This "membrane" region is characterized by displacements that are independent of the thickness coordinate and such that the mechanical strain is essentially balanced by the thermal strain. The accompanying in-plane stress components are proportional to the difference between the thermal strain and the thickness-average thermal strain.

In order to compare the solution in the "thin-shell" layer, we must first identify the "thin-shell" middle surface meridional displacement components $\tilde{V}_0^*$, $\tilde{V}_1^*$ where

$$\tilde{V}_i^* = \tilde{v}_i(\tilde{\theta}, \bar{z} = 1/2) = \tilde{V}_i - \tilde{U}_i'/2 \qquad (i = 0, 1)$$

and the thin-shell force and moment stress resultants given by

$$\tilde{N}_\theta^{(0)} = \int_0^1 \tilde{\sigma}_\theta^{(1)} \, d\bar{z}$$

$$(\tilde{N}_\phi^{(0)}, \tilde{N}_\phi^{(1)}) = \int_0^1 (\tilde{\sigma}_\phi^{(0)}, \tilde{\sigma}_\phi^{(1)})\, d\bar{z}$$

$$(\tilde{M}_\theta^{(0)}, \tilde{M}_\theta^{(1)}) = \int_0^1 (\tilde{\sigma}_\theta^{(0)}, \tilde{\sigma}_\theta^{(1)})(\bar{z}-1/2)\, d\bar{z}$$

$$(\tilde{M}_\phi^{(0)}, \tilde{M}_\phi^{(1)}) = \int_0^1 (\tilde{\sigma}_\phi^{(0)}, \tilde{\sigma}_\phi^{(1)})(\bar{z}-1/2)\, d\bar{z}$$

$$(\tilde{Q}_0, \tilde{Q}_1) = \int_0^1 (\tilde{\tau}_0, \tilde{\tau}_1)\, d\bar{z}.$$

Note that the $(1-\zeta/R)$-term is absent from the integrands of the stress resultants, where $\zeta(=z-h/2)$ is the coordinate measured normal to the middle surface. This is consistent with our intention here of comparing terms only to order $(h/a)^{1/2}$ in comparison to unity. It should also be noted that $\tilde{N}_\theta^{(0)}$ is so superscrited as it is the first non-zero force stress resultant associated with $\tilde{\sigma}_\theta$.

With the stress resultants now defined, we proceed to determine the corresponding stress resultant-displacement equations implied by the solution of the thin-shell boundary layer equations. Thus, with the stress components given by eqns (31), (32), (38), (39) and using eqns (34) and (41) to express the constants B_0, B_1 in terms of the displacements, we find that

$$\tilde{N}_\phi^{(0)} = \tilde{U}_0 - \Theta_m(\theta_0)$$

$$(1-\nu^2)(N_\theta^{(0)}, N_\phi^{(1)}) = (1+\nu)\left[\tilde{U}_1 - \tilde{\theta}\frac{d\Theta_m}{d\theta}(\theta_0)\right] + (1,\nu)\tilde{V}_1^{*\prime} + (\nu,1)\tilde{V}_0^{*}\cot\theta_0$$

$$(\tilde{M}_\theta^{(0)}, \tilde{M}_\phi^{(0)}) = \bar{M}_T - (1,\nu)\tilde{U}_0''/(12(1-\nu^2))$$

$$(\tilde{M}_\theta^{(1)}, \tilde{M}_\phi^{(1)}) = \tilde{\theta}\frac{d\bar{M}_T}{d\theta}(\theta_0) - [(1,\nu)\tilde{U}_1'' + (\nu,1)\tilde{U}_0'\cot\theta_0]/(12(1-\nu^2))$$

$$\tilde{Q}_0 = -\tilde{U}_0'''/(12(1-\nu^2))$$

$$\tilde{Q}_1 = -(\tilde{U}_1''' + \tilde{U}_0''\cot\theta_0)/(12(1-\nu^2)) + \frac{d\bar{M}_T}{d\theta}(\theta_0)$$

where

$$-(1-\nu)\bar{M}_T(\theta) = \int_0^1 \Theta(\theta,\bar{z})(\bar{z}-1/2)\, d\bar{z} = \Theta_m(\theta)/2 - \Xi_m(\theta).$$

As can readily be verified, the first four equations relating the in-plane force and moment stress resultants to the displacements are essentially the constitutive equations and are identical to those obtained in [4] from the thin-shell equations. The latter two equations are essentially the equations of moment equilibrium and are also identical to those obtained in [4]. A further interesting result can be obtained by differentiating these latter two equations. If we use eqns (36) and (42) to eliminate $\tilde{U}_0^{IV}$, $\tilde{U}_1^{IV}$ and eqns (34) and (41) to define $\tilde{V}_0'$, $\tilde{V}_1'$, we obtain the result that

$$[\tilde{V}_0^* + (1+\nu)\tilde{Q}_0]' = 0 \qquad [\tilde{V}_1^* + (1+\nu)\tilde{Q}_1]' = A_0\cos\theta_0.$$

Note that these equations for the determination of $\tilde{V}_0^*$, $\tilde{V}_1^*$, as well as eqns (36) and (42) for the determination of $\tilde{U}_0$, $\tilde{U}_1$, are identical to those obtained in [4]. Thus, the set of equations comprising the constitutive equations and the equations of equilibrium are identical with those obtained from the thin-shell equations.

It is not surprising then, to find that the corresponding expressions for the stress components in terms of the force and moment stress resultants are also identical with those obtained in [4] from the thin-shell equations. In particular, it can be shown that

$$(\tilde{\sigma}_\theta{}^{(0)}, \tilde{\sigma}_\phi{}^{(0)}) = [\Theta_m(\theta_0) - (\theta_0, \bar{z})]/(1-\nu) + (0, \tilde{N}_\phi{}^{(0)}) + 12(\bar{z} - 1/2)[(\tilde{M}_\theta{}^{(0)}, \tilde{M}_\phi{}^{(0)}) - \tilde{M}_T]$$

$$(\tilde{\sigma}_\theta{}^{(1)}, \tilde{\sigma}_\phi{}^{(1)} = \tilde{\theta}\left[\frac{d\Theta_m}{d\theta}(\theta_0) - \frac{\partial\Theta}{\partial\theta}(\theta_0, \bar{z})\right] \Big/ (1-\nu) + (\tilde{N}_\theta{}^{(0)}, \tilde{N}_\theta{}^{(1)})$$

$$+ 12(\bar{z} - 1/2)\left[(\tilde{M}_\theta{}^{(1)}, \tilde{M}_\phi{}^{(1)}) - \tilde{\theta}\frac{d\tilde{M}_T}{d\theta}(\theta_0)\right].$$

Thus, we see that the stress components are, with the exception of the temperature integrals, linear functions of the thickness ($\bar{z}$) coordinate. This result confirms the validity of the plane stress and kinematic displacement assumptions that are usually made *a priori* in developing thin-shell equations.

With the establishment of the thin-shell equations as limiting forms of the equations of Elasticity, we now use the stress-resultant concepts of thin-shell analysis to interpret the boundary conditions derived above from the requirement that the solution in the "thin-shell" layer match that in the "Elasticity" layer. Using the definitions for $\tilde{M}_\theta{}^{(0)}$, $\tilde{M}_\theta{}^{(1)}$, $\tilde{M}_T$ given above, we see that the conditions given by eqns (71) and (74) require that $\tilde{M}_\theta{}^{(0)}(\tilde{\theta} = 0)$, $\tilde{M}_\theta{}^{(1)}(\tilde{\theta} = 0)$ both vanish. Thus, the meridional bending moment must vanish through two orders of magnitude. Further, using the definition for $\tilde{N}_\theta{}^{(0)}$ and eqn (74), we see that the condition given by eqn (73) requires that $\tilde{N}_\theta{}^{(0)}(\tilde{\theta} = 0)$ vanish. As the matching requirement leading to the definition of B_0 already insures that the integral of $\tilde{\sigma}_\theta{}^{(0)}$ vanish identically, we see that the meridional force stress resultant also vanishes through two orders of magnitude. Finally, using the definitions for $\tilde{Q}_0$, $\tilde{Q}_1$, we see that the conditions given by eqns (72) and (81) require that $\tilde{Q}_0(\tilde{\theta} = 0)$, $\tilde{Q}_1(\tilde{\theta} = 0)$ both vanish. Thus, the shear force stress resultant also vanishes through two orders of magnitude. In general, then, it can be concluded that the solution of the "thin-shell" layer equations essentially satisfies the free-edge condition at $\tilde{\theta} = 0$ that the meridional and transverse force stress resultants and the meridional bending moment stress resultant must all vanish.

Having now shown that the solution of the thin-shell equations adequately describes the stresses and displacements in a sufficiently thin shell except at the edge, we now investigate the possibility that the stress in this "Elasticity" layer, which is not predicted by thin-shell equations, might dominate that in the thin-shell layer. Should this be the case, a thorough stress analysis would always have to include an evaluation of these edge stresses, i.e. a solution of eqn (62). Unfortunately, a definitive answer to this question cannot be given as this would require a solution of eqn (62) for a general temperature distribution. Some guidance, however, can be obtained from the numerical solution presented in [3] for the case of a temperature distribution such that the right-hand-side of eqn (62) is a constant. We observe from these results that the meridional normal stress decreases monotonically from its limit at the edge of the "thin-shell" layer to a boundary value of zero at $\hat{\theta} = 0$. In addition, the transverse normal stress increases monotonically from a limit value of zero at the edge of the "thin-shell" layer to its value at the edge ($\hat{\theta} = 0$). In particular, the value of the transverse normal stress at the edge ($\hat{\theta} = 0$) is relatively small in comparison with the value of the meridional normal stress at the edge ($|\hat{\theta}| \to \infty$) of the "thin-shell" layer. Thus, neither the transverse normal stress nor the meridional normal stress are capable of generating a stress concentration. Alternatively, as the "hoop" stress in the "Elasticity" layer is approximately given by

$$\hat{\sigma}_\phi{}^{(0)} = C_0 + \Theta_m(\theta_0) - \Theta(\theta_0, \bar{z}) + \nu(\hat{\sigma}_\theta{}^{(0)} + \hat{\sigma}_r{}^{(0)})$$

it is apparent from the comments made above that σ_ϕ will take on its maximum value at the edge of the thin-shell layer and then decrease slightly as $\hat{\sigma}_\theta{}^{(0)}$ decreases and $\hat{\sigma}_r{}^{(0)}$ (which is numerically smaller) increases toward $\hat{\theta} = 0$. We conclude, therefore, that the "Elasticity" layer is not a region of stress concentration, and that the dominant stresses will occur in the "thin-shell" layer. As these stresses are adequately predicted by thin-shell theory, it is apparent that there is no need to look to the equations of Elasticity for a more refined description of behavior for temperature distributions that do not depart appreciably from one of constant curvature. There still remains unanswered, however, the question of whether a stress concentration could result from a more general temperature distribution. For the time being, this answer must be left to the judgement of the designer.

REFERENCES

1. J. D. Cole, *Perturbation Methods in Applied Mathematics*. Blaisdell, Waltom, Mass. (1968).
2. M. W. Johnson and E. Reissner, On the foundations of the theory of thin elastic shells. *J. Math. Phys.* **37**, 371–392 (1958).
3. H. E. Williams, Axisymmetric thermal stress in a thin circular disk by the method of matched asymptotic expansions. *J. Solids Structures* **13**, 17–38 (1977).
4. H. E. Williams, Thermal stress analysis of thin spherical shells. *A.S.M.E. Paper* 76-DET-104 (Sept. 1976).
5. W. H. Wittrick, Edge stresses in thin shells of revolution. *Rep. SM.* 253. Dept. of Supply, Research & Development Branch, Aero. Research Labs.
6. E. Reissner, A new derivation of the equations for the deformation of elastic shells. *Am. J. Math.* **63**, 177–184 (1941).
7. H. Kraus, *Thin Elastic Shells*. Wiley, New York (1967).
8. B. A. Boley and J. H. Weiner, *Theory of Thermal Stress*. Wiley, New York (1960).
9. A. I. Lur'ye, *Three Dimensional Problems of The Theory of Elasticity*. Interscience, New York (1964).
10. J. K. Knowles, On Saint-Venant's principle in the two-dimensional linear theory of elasticity. *Archive for Rational Mechanics & Analysis*, **21**, 1–22 (1966).

APPENDIX

The third order Elasticity layer

We obtain the governing equations of the third order "Elasticity" layer using the formalism described in the previous section on the "Elasticity" layer with the addition of the term $\bar{h}^2\hat{u}_4$ to the radial displacement component expansion and the term $\bar{h}^2\hat{v}_3$ to the meridional displacement component expansion. The resulting error then becomes $0(\bar{h}^{5/2})$ for both series. Thus, proceeding as before, we obtain the two additional equations of equilibrium

$$\frac{\partial\hat{\sigma}_r^{(2)}}{\partial\bar{z}}+\frac{\partial\hat{\tau}_2}{\partial\hat{\theta}}+2\hat{\sigma}_r^{(0)}-\hat{\sigma}_\theta^{(0)}-\hat{\sigma}_\phi^{(0)}+\hat{\tau}_0\cot\theta_0-\bar{z}\frac{\partial\hat{\tau}_0}{\partial\hat{\theta}}=0 \tag{A1}$$

$$\frac{\partial\hat{\tau}_2}{\partial\bar{z}}+\frac{\partial\hat{\sigma}_\theta^{(2)}}{\partial\hat{\theta}}+(\hat{\sigma}_\theta^{(0)}-\hat{\sigma}_\phi^{(0)})\cot\theta_0+3\hat{\tau}_0-\bar{z}\frac{\partial\hat{\sigma}_\theta^{(0)}}{\partial\hat{\theta}}=0 \tag{A2}$$

and four stress-displacement equations

$$\frac{\partial\hat{u}_4}{\partial\bar{z}}=\hat{\sigma}_r^{(2)}-\nu(\hat{\sigma}_\theta^{(2)}+\hat{\sigma}_\phi^{(2)})+\hat{\theta}\frac{\partial\Theta}{\partial\theta}(\theta_0,\bar{z}) \tag{A3}$$

$$\hat{u}_2+\frac{\partial\hat{v}_3}{\partial\hat{\theta}}=\hat{\sigma}_\theta^{(2)}-\nu(\hat{\sigma}_r^{(2)}+\hat{\sigma}_\phi^{(2)})+\hat{\theta}\frac{\partial\Theta}{\partial\theta}(\theta_0,\bar{z})+\bar{z}[\hat{\sigma}_\theta^{(0)}-\nu(\hat{\sigma}_r^{(0)}+\hat{\sigma}_\phi^{(0)})+\Theta(\theta_0,\bar{z})] \tag{A4}$$

$$\frac{\partial\hat{v}_3}{\partial\bar{z}}+\frac{\partial\hat{u}_4}{\partial\hat{\theta}}-\bar{z}\frac{\partial\hat{u}_2}{\partial\hat{\theta}}-\hat{v}_1=2(1+\nu)\hat{\tau}_2 \tag{A5}$$

$$\hat{u}_2+\hat{v}_1\cot\theta_0=\hat{\sigma}_\phi^{(2)}-\nu(\hat{\sigma}_\theta^{(2)}+\hat{\sigma}_r^{(2)})+\hat{\theta}\frac{\partial\Theta}{\partial\theta}(\theta_0,\bar{z})+\bar{z}\,[\hat{\sigma}_\phi^{(0)}-\nu(\hat{\sigma}_r^{(0)}+\hat{\sigma}_\theta^{(0)})+\Theta(\theta_0,\bar{z})]. \tag{A6}$$

As the solution can be most conveniently be formulated in terms of the stresses $\hat{\sigma}_r^{(2)}$, $\hat{\sigma}_\theta^{(2)}$, $\hat{\tau}_2$ to form an equivalent plane strain problem, we proceed to find a compatibility equation to augment the equations of equilibrium and the eqn (A6) for $\hat{\sigma}_\phi^{(2)}$. Thus, if we eliminate $\hat{u}_4$, $\hat{v}_3$ from eqns (A3)–(A5) and express $\hat{u}_2$, $\hat{v}_1$ in terms of the first order stresses using eqns (47), (48), (51), (57), we obtain the following equation of compatibility

$$\begin{aligned}(1+\nu)&\left(\frac{\partial^2\hat{\sigma}_r^{(2)}}{\partial\hat{\theta}^2}+\frac{\partial^2\hat{\sigma}_\theta^{(2)}}{\partial\bar{z}^2}\right)-\nu\nabla^2(\hat{\sigma}_r^{(2)}+\hat{\sigma}_\phi^{(2)}+\hat{\sigma}_\theta^{(2)})-2(1+\nu)\frac{\partial^2\hat{\tau}_2}{\partial\hat{\theta}\partial\bar{z}}\\ &=-\hat{\theta}\frac{\partial^3\Theta}{\partial\theta\partial\bar{z}^2}(\theta_0,\bar{z})-\frac{\partial}{\partial\bar{z}}\left[\bar{z}\frac{\partial\Theta}{\partial\bar{z}}(\theta_0,\bar{z})\right]+\bar{z}\left\{(1+\nu)\left(\frac{\partial^2\hat{\sigma}_r^{(0)}}{\partial\hat{\theta}^2}-\frac{\partial^2\hat{\sigma}_\theta^{(0)}}{\partial\bar{z}^2}\right)\right.\\ &\left.-\nu\left(\frac{\partial^2}{\partial\hat{\theta}^2}-\frac{\partial^2}{\partial\bar{z}^2}\right)(\hat{\sigma}_r^{(0)}+\hat{\sigma}_\phi^{(0)}+\hat{\sigma}_\theta^{(0)})\right\}-\frac{\partial}{\partial\bar{z}}[(2+\nu)\hat{\sigma}_\theta^{(0)}-(1+2\nu)\hat{\sigma}_r^{(0)}-\nu\hat{\sigma}_\phi^{(0)}]+2(1+\nu)\frac{\partial\hat{\tau}_0}{\partial\hat{\theta}}.\end{aligned}$$

Once more, we note that this equation is simply the third order form of the compatibility equation ($Q_{33}=0$) proposed by Lur'ye. The final form of the required compatibility equation is obtained by eliminating $\hat{\sigma}_\phi^{(2)}$, $\hat{\sigma}_\phi^{(0)}$ using eqn (A6), 53 respectively with the result that

$$\begin{aligned}\frac{\partial^2\hat{\sigma}_r^{(2)}}{\partial\hat{\theta}^2}&+\frac{\partial^2\hat{\sigma}_\theta^{(2)}}{\partial\bar{z}^2}-\nu\nabla^2(\hat{\sigma}_r^{(2)}+\hat{\sigma}_\theta^{(2)})-2\frac{\partial^2\hat{\tau}_2}{\partial\hat{\theta}\partial\bar{z}}\\ &=-\hat{\theta}\frac{\partial^3\Theta}{\partial\theta\partial\bar{z}^2}(\theta_0,\bar{z})-\bar{z}\frac{\partial^2\Theta}{\partial\bar{z}^2}(\theta_0,\bar{z})-\frac{\partial\Theta}{\partial\bar{z}}(\theta_0,\bar{z})+\bar{z}\left[\frac{\partial^2\hat{\sigma}_r^{(0)}}{\partial\hat{\theta}^2}-\frac{\partial^2\hat{\sigma}_\theta^{(0)}}{\partial\bar{z}^2}\right.\\ &\left.-\nu\left(\frac{\partial^2}{\partial\hat{\theta}^2}-\frac{\partial^2}{\partial\bar{z}^2}\right)(\hat{\sigma}_r^{(0)}+\hat{\sigma}_\theta^{(0)})\right]-\nu\frac{\partial}{\partial\hat{\theta}}(\hat{\sigma}_\theta^{(0)}+\hat{\sigma}_r^{(0)})\cot\theta_0-\frac{\partial}{\partial\bar{z}}(\hat{\sigma}_r^{(0)}+2\hat{\sigma}_\theta^{(0)})\end{aligned} \tag{A7}$$

where the right hand side has been simplified using the equations of equilibrium eqns (45) and (46).

Returning now to the equations of equilibrium eqns (A1) and (A2), it follows on introducing the definitions of the potential $\phi^{(0)}$ that they can be written in the form

$$\frac{\partial R}{\partial \bar{z}} + \frac{\partial T}{\partial \hat{\theta}} = \hat{A}_0 - \Theta(\theta_0, \bar{z}) \tag{A8}$$

$$\frac{\partial S}{\partial \hat{\theta}} + \frac{\partial T}{\partial \bar{z}} = [\hat{A}_0 - \Theta(\theta_0, \bar{z})] \cot \theta_0 \tag{A9}$$

where

$$R = \hat{\sigma}_r^{(2)} + 2\bar{z}\hat{\sigma}_r^{(0)} - (1+\nu)\frac{\partial \phi^{(0)}}{\partial \bar{z}} - (2-\nu)\frac{\partial \phi^{(0)}}{\partial \hat{\theta}} \cot \theta_0 \tag{A10}$$

$$S = \hat{\sigma}_\theta^{(2)} - \nu \frac{\partial \phi^{(0)}}{\partial \hat{\theta}} \cot \theta_0 - (2-\nu)\frac{\partial \phi^{(0)}}{\partial \bar{z}} \tag{A11}$$

$$T = \hat{\tau}_2 + \bar{z}\hat{\tau}_0 - \nu \frac{\partial \phi^{(0)}}{\partial \hat{\theta}} + (1-\nu)\frac{\partial \phi^{(0)}}{\partial \bar{z}} \cot \theta_0. \tag{A12}$$

The boundary conditions on R, S, T follow from those required of $\phi^{(0)}$ and imply that

$$R = T = 0 \quad \text{on} \quad \bar{z} = 0, 1; \quad -\infty < \hat{\theta} \leq 0 \tag{A13}$$

$$S = T = 0 \quad \text{on} \quad \hat{\theta} = 0; \quad 0 \leq \bar{z} \leq 1. \tag{A14}$$

Now, proceeding as in [3], we construct a particular solution of eqns (A8) and (A9) by assuming that S is a linear function of $\bar{z}$, i.e.

$$S_{\text{part}} = f(\hat{\theta}) + \bar{z} \cdot g(\hat{\theta})$$

and determining f, g such that eqns (A8) and (A9) and the boundary conditions eqns (A13) are satisfied. The associated particular solutions for T, R are obtained by integrating eqns (A8) and (A9) from $\bar{z} = 0$ to $\bar{z}$-arbitrary using the boundary conditions on $\bar{z} = 0$ to evaluate the constants of integration. With the form of T_{part}, R_{part} now determined in terms of the derivatives of f, g, we obtain the equations governing f, g by imposing the boundary conditions at $\bar{z} = 1$ on the expressions for T_{part}, R_{part}. At this stage, the functions f, g can be determined in terms of four constants of integration. If the constants of integration are chosen such that $f(0) = g(0) = 0$, and we also require that

$$\int_0^1 T(\hat{\theta} = 0, x)\,\mathrm{d}x = 0$$

it can be shown that

$$R_{\text{part}} = \hat{A}_0\bar{z} - \Xi(\theta_0, \bar{z}) - \bar{z}^2(3 - 2z)[\hat{A}_0 - \Theta_m(\theta_0)] \tag{A15}$$

$$S_{\text{part}} = \hat{\theta}\{\hat{A}_0 - \Theta_m(\theta_0) + 3(1-2\bar{z})[\Theta_m(\theta_0) - 2\Xi_m(\theta_0)]\} \cot \theta_0 - 3\hat{\theta}^2(1-2\bar{z})[\hat{A}_0 - \Theta_m(\theta_0)] \tag{A16}$$

$$T_{\text{part}} = \bar{z}(1-\bar{z})\{2[3\Xi_m(\theta_0) - \Theta_m(\theta_0)] \cot \theta_0 + 6\hat{\theta}[\hat{A}_0 - \Theta_m(\theta_0)]\} + [\bar{z}^2\Theta_m(\theta_0) - \Xi(\theta_0, \bar{z})] \cot \theta_0. \tag{A17}$$

With the particular solution of eqns (A8) and (A9) now known, we can proceed to formulate the third order "Elasticity" layer problem as an equivalent plane strain problem. This is accomplished by defining the functions R^*, S^*, T^* by

$$R = R^* + R_{\text{part}} \qquad S = S^* + S_{\text{part}} \qquad T = T^* + T_{\text{part}}$$

where R^*, S^*, T^* satisfy the homogeneous forms of eqns (A8) and (A9), and hence can be expressed in terms of an Airy stress function in the form

$$R^* = \frac{\partial^2 \phi^*}{\partial \hat{\theta}^2} \qquad S^* = \frac{\partial^2 \phi^*}{\partial \bar{z}^2} \qquad T^* = -\frac{\partial^2 \phi^*}{\partial \hat{\theta} \partial \bar{z}}.$$

Finally, the equation governing ϕ^* is obtained by substituting eqns (A10)–(A12), (A15)–(A18) into eqn (A7) with the result that

$$\nabla^4 \phi^* = -\frac{\hat{\theta}}{1-\nu} \cdot \frac{\partial^3 \Theta}{\partial \theta \partial \bar{z}^2}(\theta_0, \bar{z}) + 12(1-2\bar{z})[\hat{A}_0 - \Theta_m(\theta_0)] - \frac{1+\nu}{1-\nu} \cdot \frac{\partial \Theta}{\partial \bar{z}}(\theta, \bar{z})$$
$$- 4\bar{z}\frac{\partial^2}{\partial \bar{z}^2}\left[\nabla^2 \phi^{(0)} + \frac{\Theta(\theta_0, \bar{z})}{1-\nu}\right] - 2\frac{\partial(\nabla^2 \phi^{(0)})}{\partial \hat{\theta}} \cot \theta_0 - 4\frac{\partial^3 \phi^{(0)}}{\partial \bar{z}^3}. \tag{A18}$$

The stress-free boundary conditions can also be expressed in terms of ϕ^*, and lead to the following equivalent representation

$$\phi^* = 0 \qquad \frac{\partial \phi^*}{\partial \bar{z}} = 0 \quad \text{on} \quad \bar{z} = 0, 1 \quad (-\infty < \hat{\theta} \leq 0) \tag{A19}$$

$$\phi^* = 0 \qquad \frac{\partial \phi^*}{\partial \hat{\theta}} = \left\{ \bar{z}^2(3-2\bar{z})[3\Xi_m(\theta_0) - \Theta_m(\theta_0)] + \bar{z}^3\Theta_m(\theta_0) - 3\int_0^{\bar{z}} \Xi(\theta_0, x)\,dx \right\} \frac{\cot\theta_0}{3}$$

$$\text{on} \quad \hat{\theta} = 0 \quad (0 \leq \bar{z} \leq 1). \tag{A20}$$

Thus, the problem of determining the stresses in the third order "Elasticity" layer is reduced to an equivalent plane strain problem and is analogous to the problem of determining the transverse displacements of a laterally loaded rectangular plate. However, in this case, the plate is clamped along the sides $\bar{z} = 0, 1$, but hinged on $\hat{\theta} = 0$ such that the displacement is zero and the slope is prescribed.

As the present requirements are simply to obtain the limiting forms of the stress components $\hat{\sigma}_r^{(2)}$, $\hat{\tau}_2$, we need not obtain the complete solution of eqn (A18), but only the limiting form as it contributes to the required stresses. Clearly, the limiting form of ϕ^* is determined by the limiting form of the right hand side of eqn (A18). In particular, we note that all the terms with the exception of the first, are either constants, functions of $\bar{z}$, transcendentally small or become functions of $\bar{z}$ as $|\hat{\theta}| \to \infty$. Thus, the required limiting form of ϕ^* is the sum of a function of $\bar{z}$ and a particular solution based on the first term and given by

$$\phi^*_{\text{part}} = -\frac{\hat{\theta}}{1-\nu}\frac{\partial}{\partial \theta}\left\{ \int_0^{\bar{z}} \Xi(\theta, x)\,dx + \bar{z}^2(1-\bar{z})\Theta_m(\theta) - \bar{z}^2(3-2\bar{z})\Xi_m(\theta) \right\}\Bigg|_{\theta=\theta_0}.$$

Clearly, this is the only term which will contribute to T^*, and leads to the conclusion that R^* is transcendentally small as $|\hat{\theta}| \to \infty$.

The limiting form of $\hat{\tau}_2$ is obtained from eqn (A12) and the observations (see eqn 68) that $\hat{\tau}_0$, $(\partial\phi^{(0)}/\partial\hat{\theta})$ are transcentendally small as $|\hat{\theta}| \to \infty$. Thus, with T_{part} given by eqn (A17), $\phi^{(0)} \approx \phi^{(0)}_{\text{part}}$ and $\phi^* \approx \phi^*_{\text{part}}$, it follows that the required limiting form of $\hat{\tau}_2$ is

$$\hat{\tau}_2 \approx 6\hat{\theta}\bar{z}(1-\bar{z})[\hat{A}_0 - \Theta_m(\theta_0)] + \left\{ \frac{\partial \Xi}{\partial \theta}(\theta_0, \bar{z}) - \bar{z}(3\bar{z}-2)\frac{d\Theta_m}{d\theta}(\theta_0) - 6\bar{z}(1-\bar{z})\frac{d\Xi_m}{d\theta}(\theta_0) \right\} \Big/ (1-\nu). \tag{A21}$$

Finally, the required limiting form of $\hat{\sigma}_r^{(2)}$ is obtained from eqn (A10) and the observations (see eqn 68) that $\hat{\sigma}_r^{(0)}$, $(\partial\phi^{(0)}/\partial\hat{\theta})$ are transcendentally small as $|\hat{\theta}| \to \infty$. Thus, with R_{part} given by eqn (A15), $\phi^{(0)} \approx \phi^{(0)}_{\text{part}}$, it follows that the required limiting form of $\hat{\sigma}_r^{(2)}$ is

$$\hat{\sigma}_r^{(2)} \approx \{-2\Xi(\theta_0, \bar{z}) - 2\bar{z}\Theta_m(\theta_0)[\bar{z}^2 - 3\bar{z} + 1 + \nu(1-\bar{z}^2)]\}/(1-\nu) + \bar{z}(1-\bar{z})\left[6\frac{1+\nu}{1-\nu}\Xi_m(\theta_0) + \hat{A}_0(1-2\bar{z})\right]. \tag{A22}$$